Nonlinear Wave Interactions in Rotating Shallow Water Equations on the Sphere

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joint work with **Marco Dourado**, **Breno Raphaldini, Andre Teruya**

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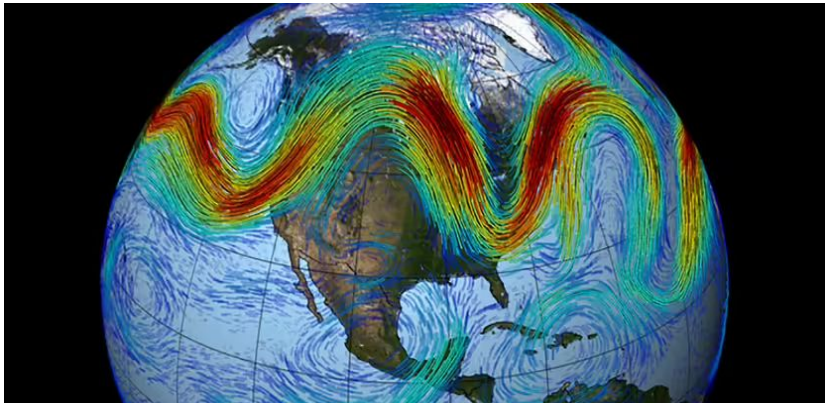
Visiting professor - MOX - Politecnico di Milano (2026)



WCCM-ECCOMAS - July - 2026

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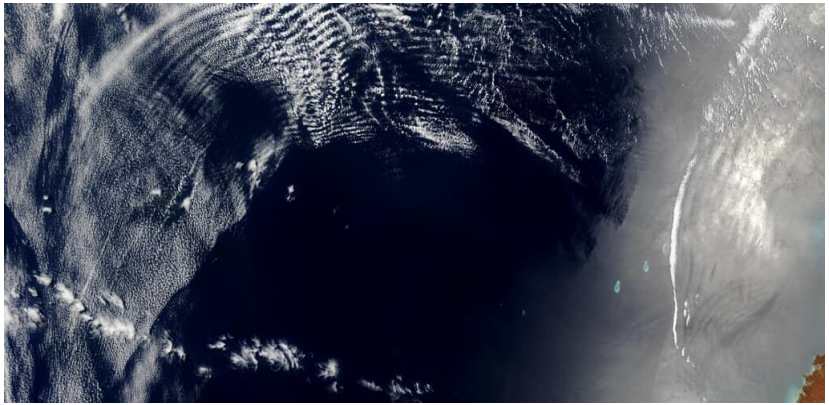
Atmospheric Wave Dynamics



Rossby-Haurwitz Waves

Slow dynamics governed by potential vorticity.

The jet stream interacting with a Rossby Wave.
NASA/GSFC



Inertio-gravity Waves

Fast dynamics driven by gravity and buoyancy.

Terra satellite capturing gravity waves in clouds in Indian Ocean.
NASA/GSFC

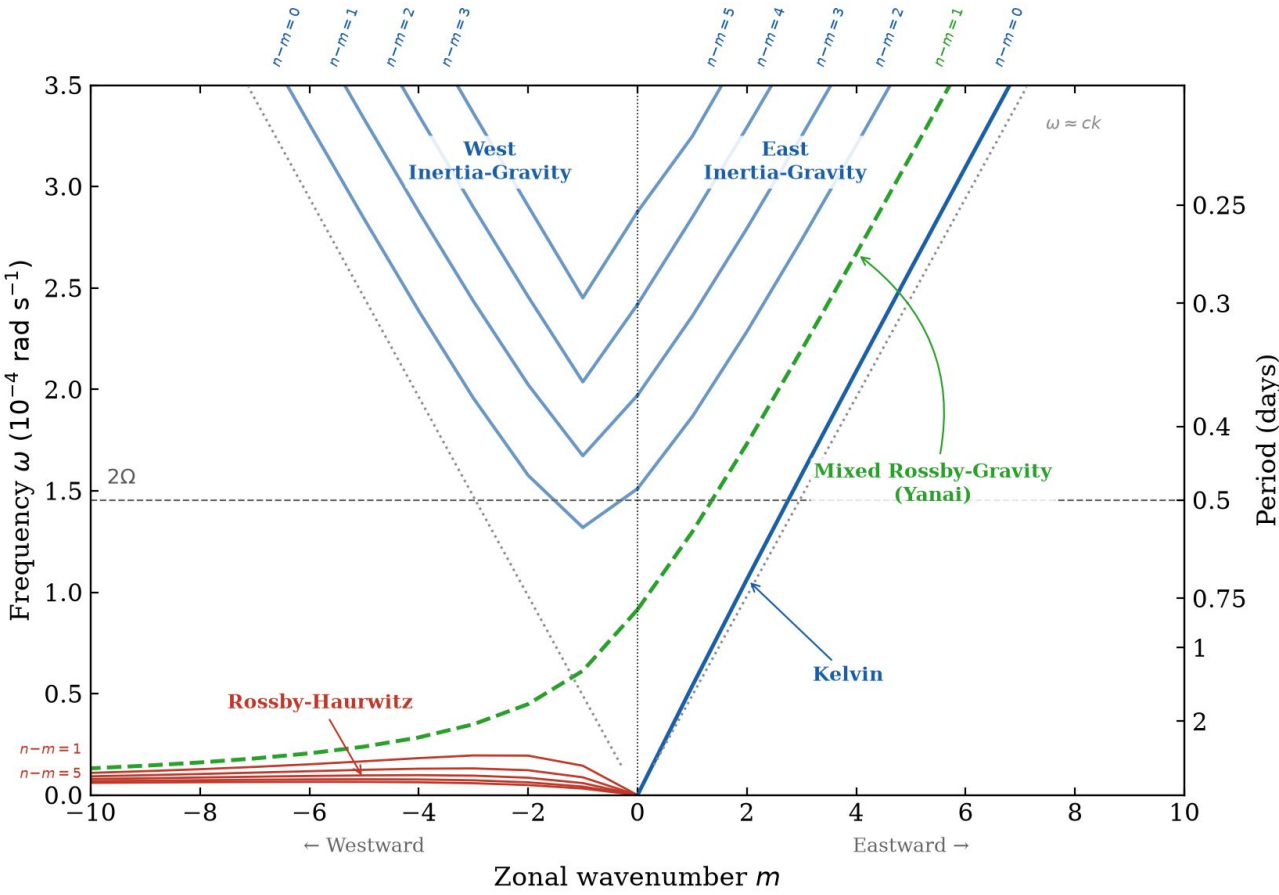
Wave Dynamics - Linear

Linear Horizontal Dynamics
(on the sphere)

Fast

Multiscale
Independent

Slow

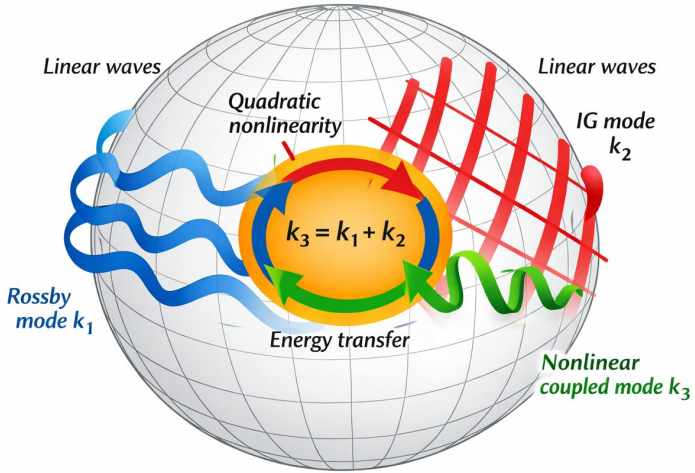


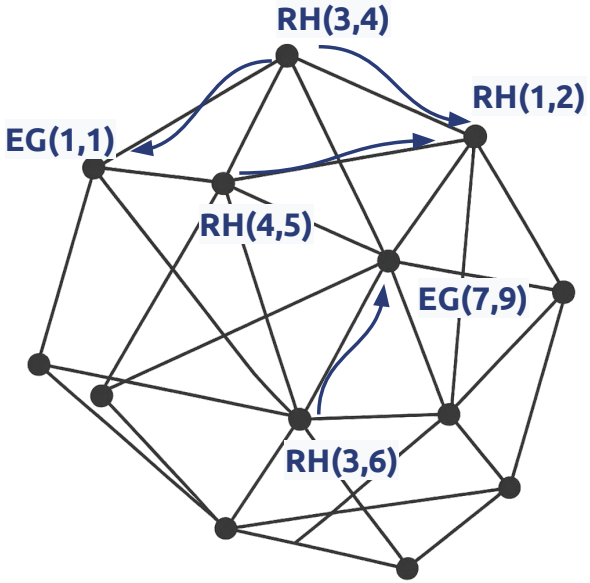
Linear Rotating Shallow Water Eigenvalues (Hough modes) on the sphere with 10km equivalent depth

Wave dynamics -> Nonlinear coupling

Nonlinearities break the independence
Energy exchanges, Waves coupling

$$(\partial_t + L)\,\vec{u} = \mathcal{N}(\vec{u}, \vec{u})$$





U. Achatz. *The Dynamics of the Shallow-Water Equations.* In: Atmospheric Dynamics. Springer Spektrum, Berlin, Heidelberg, 2022.
A. Madja. *Introduction to PDEs and Waves for the Atmosphere and Ocean.* American Mathematical Society, 2000.

Computational Relevance of Nonlinear interaction

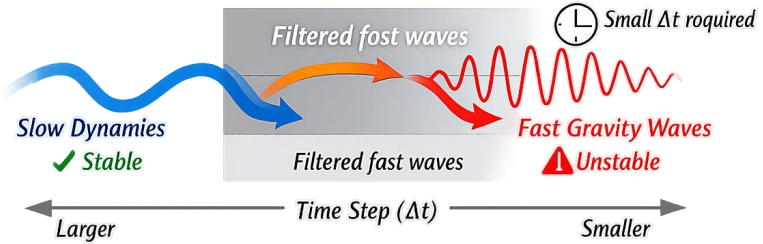
Numerical Models

Efficiency

Fast gravity waves => smaller **time-step** (stability in explicit numerical models)

*Often these waves are **filtered or attenuated out** to allow for larger steps in slow dynamics studies.*

Scale separation: nonlinear coupling breaks comom scale separation assumptions in numerical models.



Risk of losing slow frequency waves originated from fast nonlinear coupling

Durran, D. R. (2010). Numerical methods for fluid dynamics: With applications to geophysics (Vol. 32). Springer Science & Business Media.
R. Salmon (1998). *Lectures on Geophysical Fluid Dynamics.* Oxford.

Impacts on Fast-Slow Nonlinear Wave Interactions

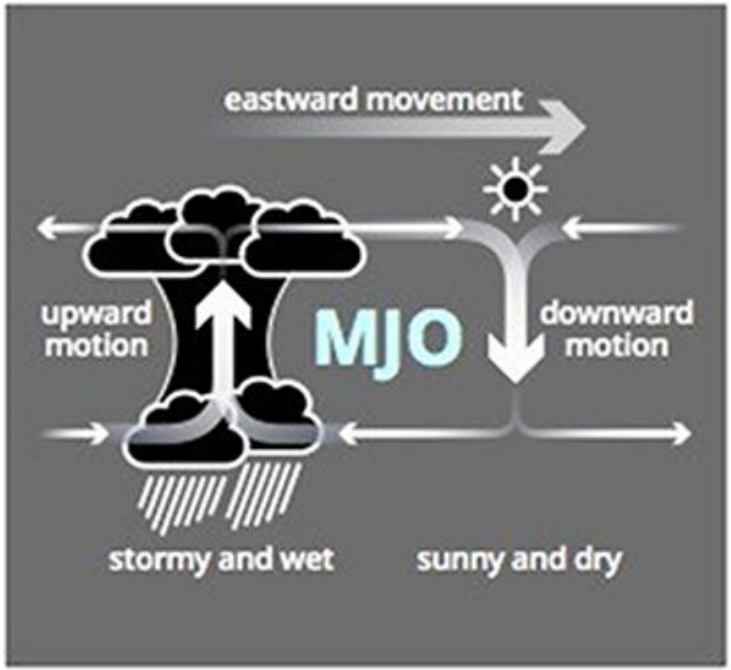
Seasonal/Climate phenomena

Madden Julian Oscillation

A key driver of tropical weather variability, influencing:

- Global **rainfall** patterns
- **Genesis** of tropical cyclones
- Interactions with **El-Niño** cycles

*Do **fast waves affects these slow** evolving phenomena?*



Low frequency (weeks+) phenomena usually cannot be explained by linear theory

Yoneyama, K., & Zhang, C. (2020). Years of the maritime continent. Geophysical Research Letters, 47(12), e2020GL087182.
C. Zhang. *Madden-julian oscillation.* Rev. Geophys., 2005.

Background: Maths and Some Key Papers

On the plane

Embid and Majda (1996)

Shallow Water Eq., f-plane and derive the averaging principle

Vortical modes move independently of gravity modes; solutions in quasi-geostrophic regime.

Raupp and Dias (2005)

Equatorial-plane Shallow Water Equations

Energy redistribution between Rossby/gravity waves. High frequency affects interseasonal dynamics.

On the sphere

Raphaldini et al. (2022)

Barotropic nondivergent model on the sphere

Rossby-Haurwitz waves & Precession resonance. Nonlinear contributions to low frequency.

Schubert et al (2008) / Verkley (2009)

Quasi-geostrophic model on the sphere theory

Good approximation for slow dynamics of the Shallow Water model.

Vannest and Vial (1994)

Slowly rotating sphere

Analytical solutions for Rossby-Haurwitz (rotational) and Gravity (divergent) interactions.

Open Problem

Nonlinear interactions analyses for the

Shallow Water Equations on the Sphere

Rotating Shallow Water Model on the Sphere

$$\frac{Du}{Dt} - \left(2\Omega \sin\phi + \frac{u \tan\phi}{a}\right) v + \frac{g}{a\cos\phi}\frac{\partial h}{\partial \lambda} = 0$$

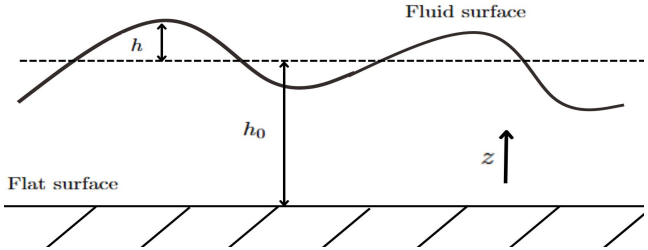
$$\frac{Dv}{Dt} + \left(2\Omega \sin\phi + \frac{u \tan\phi}{a}\right) u + \frac{g}{a}\frac{\partial h}{\partial \phi} = 0$$

$$\frac{Dh}{Dt} + \frac{h + h_0}{a\cos\phi}\left(\frac{\partial u}{\partial \lambda} + \frac{\partial (v\cos\phi)}{\partial \phi}\right) = 0$$

λ is the longitude
μ is the latitude
(u, v) is the velocity
h is the height
$f = 2\Omega \sin\phi$ is the Coriolis parameter

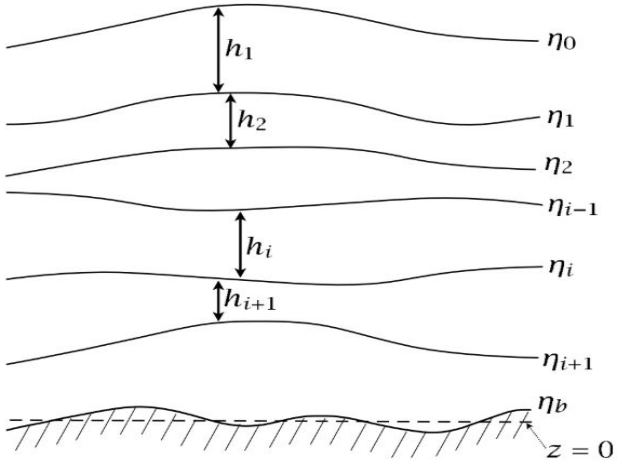
Material derivative

$$\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial \psi}{\partial \lambda} + \frac{v}{a}\frac{\partial \psi}{\partial \phi}$$



Fast (gravity) and slow (Rossby-Haurwitz) waves

Stratification obtained in two layer model
Primitive equations obtained as multi-layer SW model



Normal Modes of the RSWE

$$(\partial_t + L)\,\vec{u} = \mathcal{N}(\vec{u}, \vec{u})$$

Laplace's tidal operator (L)

$$L = \begin{pmatrix} 0 & -\sin\phi & \frac{\gamma}{\cos\phi}\partial_\lambda \\ \sin\phi & 0 & \gamma\partial_\phi \\ \frac{\gamma}{\cos\phi}\partial_\lambda & \frac{\gamma}{\cos\phi}\partial_\phi(\cos\phi(\cdot)) & 0 \end{pmatrix}$$

$$\gamma = \frac{\sqrt{gh_0}}{2\Omega a}$$

L is linear in longitude

Nonlinear Bilinear Operator (N)

$$\mathcal{N}(\vec{u_1}, \vec{u_2}) = \gamma \begin{pmatrix} -\frac{u_1}{\cos\phi}\frac{\partial u_2}{\partial\lambda} - v_1\frac{\partial u_2}{\partial\phi} + u_1 v_2 \tan\phi \\ -\frac{u_1}{\cos\phi}\frac{\partial v_2}{\partial\lambda} - v_1\frac{\partial v_2}{\partial\phi} - u_1 u_2 \tan\phi \\ -\frac{u_1}{\cos\phi}\frac{\partial h_2}{\partial\lambda} - v_1\frac{\partial h_2}{\partial\phi} - \frac{h_2}{\cos\phi}\left(\frac{\partial u_1}{\partial\lambda} + \frac{\partial(v_1\cos\phi)}{\partial\phi}\right) \end{pmatrix}$$

N is bilinear

$$\vec{u}(t,\lambda,\phi) = H(\lambda,\phi)e^{-i\omega t}$$

Normal Linear Modes

$$(L - i\omega)H(\lambda,\phi) = 0$$

$$H(\lambda,\phi) = \Theta(\phi)e^{i\lambda m} \longrightarrow \text{zonal}$$

Hough harmonics

→ meridional

(Hough function)

For each zonal wavenumber there are infinitely many meridional functions.

A. Kasahara. *Numerical integration of the global barotropic primitive equations with hough harmonic expansion.* Journal of the Atmospheric Sciences, 1977.
P. Swarztrauber and A. Kasahara. The vector harmonic analysis of laplace's tidal equations. Society for Industrial and Applied Mathematics, 1985.

Galerkin Expansion

$$\vec{u}(\lambda,\phi,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\alpha=1}^{3} A^m_{n,\alpha}(t) H^m_{n,\alpha}(\lambda,\phi) e^{-i\omega^m_{n,\alpha}t}$$

Expansion in Hough harmonics

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\alpha=1}^{3} A^m_{n,\alpha}(t)\Theta^m_{n,\alpha}(\phi) e^{i(\lambda m-\omega^m_{n,\alpha}t)}$$

$$\mathcal{N}(\vec{u},\vec{u}) = \frac{\partial \vec{u}}{\partial t} + L\vec{u} = \sum_a \frac{dA_a}{dt} H_a e^{-i\omega_a t}$$

Projecting on single mode

$$\frac{dA_a}{dt} = \left\langle \sum_a \frac{dA_a}{dt} H_a e^{-i\omega_a t}, H_a e^{-i\omega_a t} \right\rangle = \left\langle \mathcal{N}(\vec{u},\vec{u}), H_a e^{-i\omega_a t} \right\rangle$$

$$= -i\gamma \sum_b \sum_c A_b(t) A_c(t) e^{-i(\omega_b+\omega_c-\omega_a)t} \left\langle \mathcal{N}\left(\Theta_b e^{im_b\lambda}, \Theta_c e^{im_c\lambda}\right), \Theta_a(\phi) e^{im_a\lambda} \right\rangle$$

Coupling coefficient: nonlinear interaction

Constraints for interaction:

Longitude - zonal wavenumbers: $m_a = m_b + m_c$

Latitude - Symmetry constraints (wrt equator)

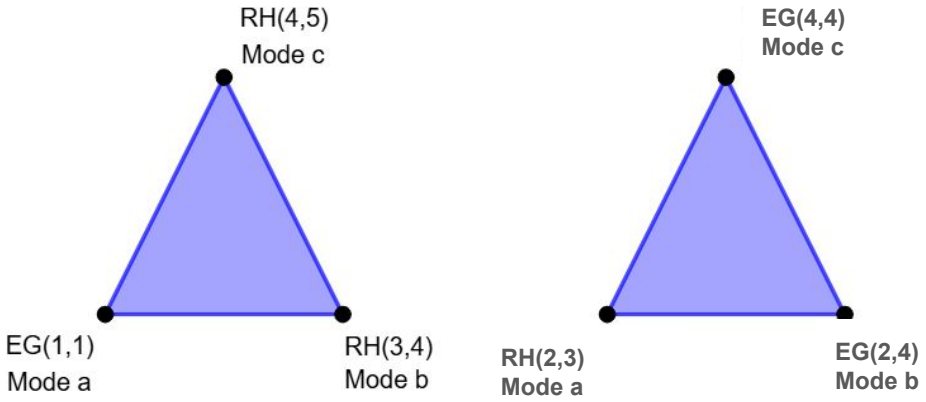
B. Raphaldini, P. Peixoto, A. Teruya, C. Raupp, and M. Bustamante. *Precession resonance of rossby wave triads and the generation of low frequency atmospheric oscillations.* Physics of Fluids, 2022.

Three waves

$$\begin{aligned}\overrightarrow{u}(\lambda,\phi,t) &= A_a(t)\Theta_a(t)e^{i(\lambda m_a-\omega_a t)} + A_b(t)\Theta_b(t)e^{i(\lambda m_b-\omega_b t)} \\ &+ A_c(t)\Theta_c(t)e^{i(\lambda m_c-\omega_c t)} + c.c.\end{aligned}$$

Change of variables

$$A'_a = A_a e^{-it\omega_a},\ A'_b = A_b e^{-it\omega_b},\ A'_c = A_c e^{-it\omega_c}$$



Three-wave system

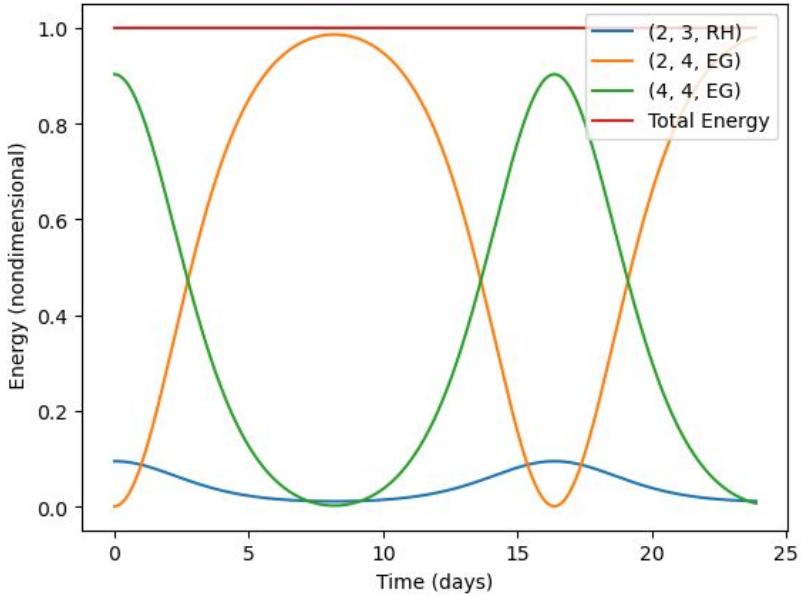
$$\begin{aligned}\frac{dA_a}{dt} &= -i\omega_a A_a + i\alpha^a_{b,c}A^*_b A_c \\ \frac{dA_b}{dt} &= -i\omega_b A_b + i\alpha^b_{a,c}A^*_a A_c \\ \frac{dA_c}{dt} &= -i\omega_c A_c + i\alpha^c_{a,b}A_a A_b\end{aligned}$$

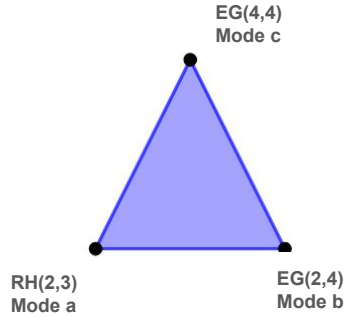
Total energy (quadratic+cubic) is conserved

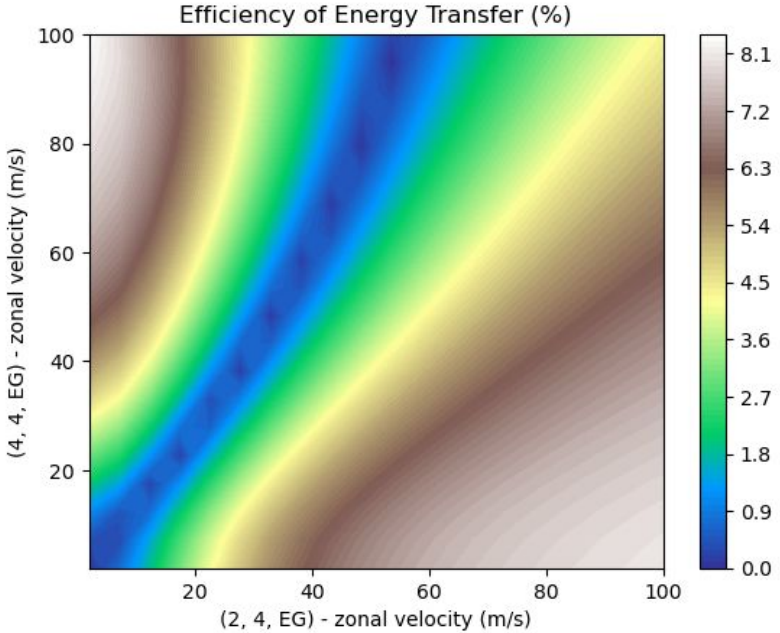
A. Craik. *Wave interactions and fluid flows.* Cambridge University Press, 1988.

East Gravity to Rossby

		Coup. coeff.	Freq.	Zonal (m/s)	Amp.
Mode a	2,3,RH	0.02868	-0.1336	30	0.0756
Mode b	2,4,EG	-0.33704	1.5694	2	0.00649
Mode c	4,4,EG	-0.30817	1.435	100	0.2332
Mismatch	0.00051				
Efficiency	8.3857 %				

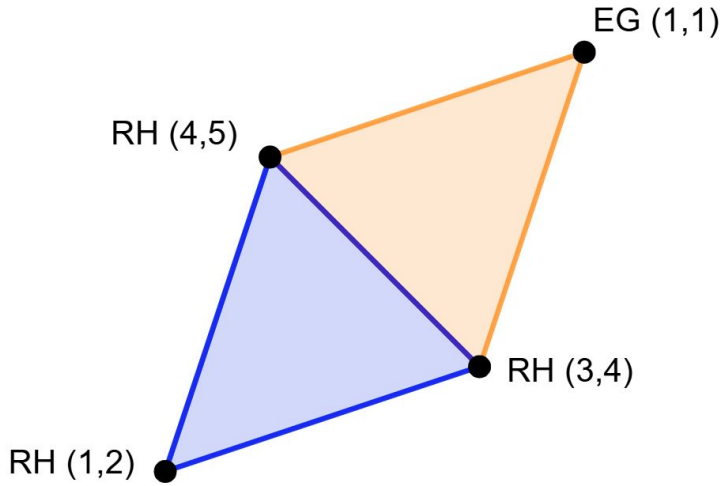






4 WAVES

$$\frac{dA_a}{dt} = -i\omega_a A_a + i\alpha^a_{b,c} A_b A_c + i\alpha^a_{b,d} A_b A_d,$$
$$\frac{dA_b}{dt} = -i\omega_b A_b + i\alpha^b_{a,c} A_a A^*_c + i\alpha^b_{a,d} A_a A^*_d,$$
$$\frac{dA_c}{dt} = -i\omega_c A_c + i\alpha^c_{a,b} A_a A^*_b,$$
$$\frac{dA_d}{dt} = -i\omega_d A_d + i\alpha^d_{a,b} A_a A^*_b,$$

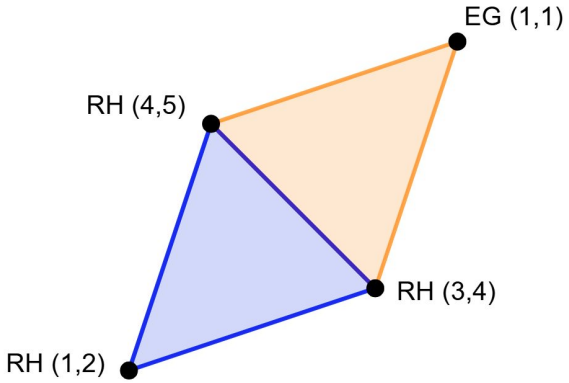


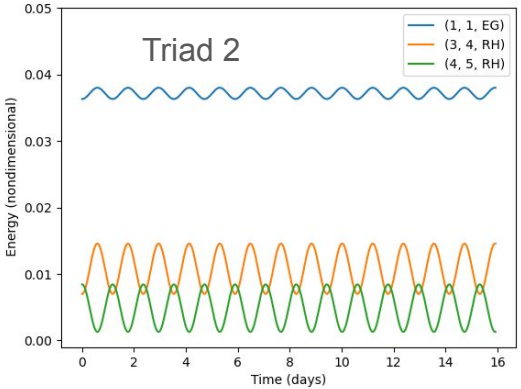
Conserve the Manley-Rowe relations:

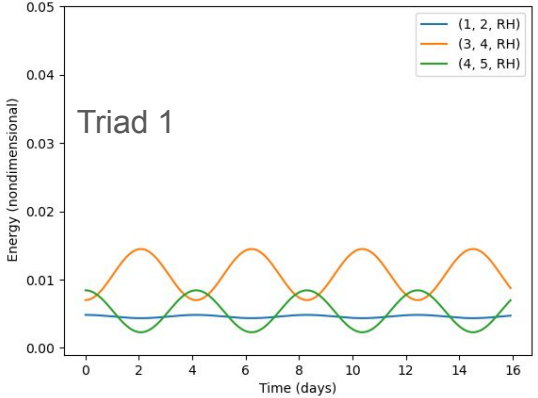
$$J_{a,c,d} = |A_a|^2 + \frac{\alpha^a_{b,c}}{\alpha^c_{a,b}}|A_c|^2 + \frac{\alpha^a_{b,d}}{\alpha^d_{a,b}}|A_d|^2,$$
$$J_{b,c,d} = |A_b|^2 - \frac{\alpha^b_{a,c}}{\alpha^c_{a,b}}|A_c|^2 - \frac{\alpha^b_{a,d}}{\alpha^d_{a,b}}|A_d|^2.$$

…but doesn't conserve total energy exactly: $O(A^4)$

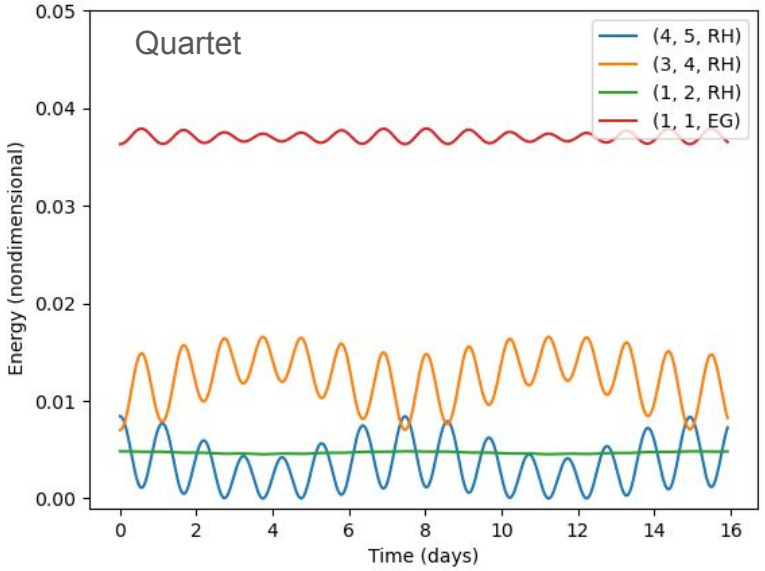
4 WAVES





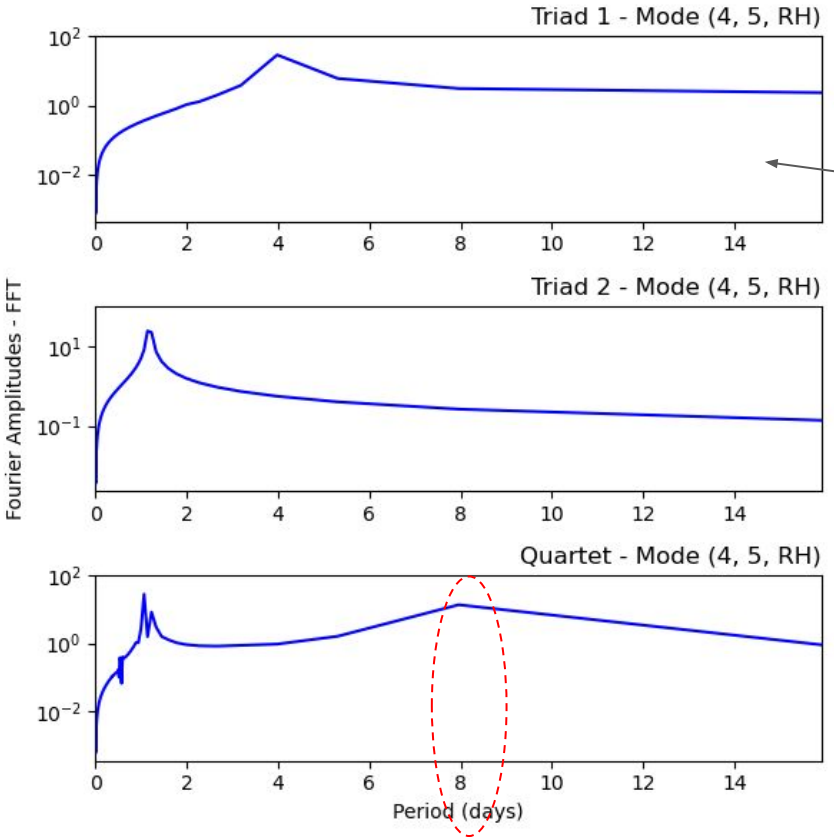


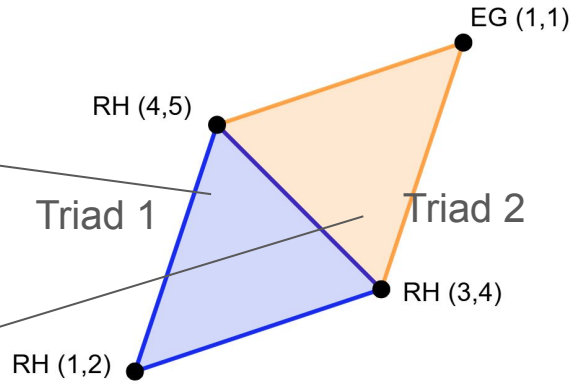
Generation of low frequency signal!



		Freq.	Coeff. 1	Coeff. 2	Zonal	Amplitude
Mode a	4,5,RH	-0.12522	0.3638	-0.5572	30	0.0918137
Mode b	3,4,RH	-0.13431	0.4434	-0.5902	30	0.0836648
Mode c	1,2,RH	-0.09934	-0.0284	-	30	0.0695306
Mode d	1,1,EG	0.36913	-	-0.1323	80	0.1905199

Exchange periods - Energy leaking to RH4,5 mode

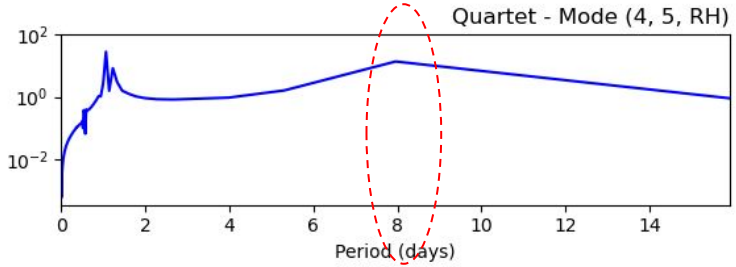




Quartet

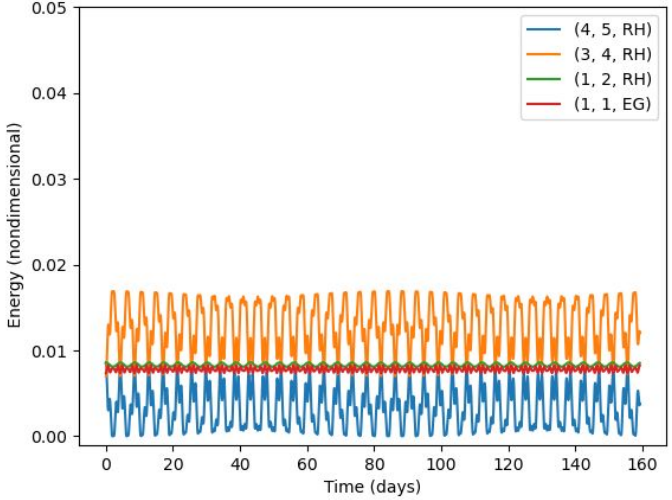
- Larger period (~ 8 days) is generated
- Fastest period remains

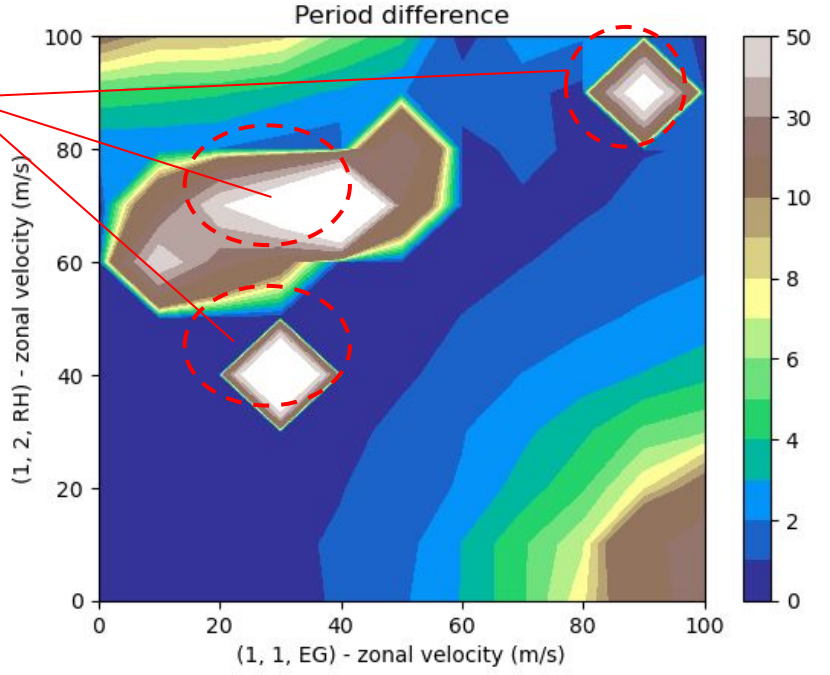
Exchange periods - Energy leaking to RH4,5 mode



Changing initial amplitudes of gravity waves

Periods larger than 50 days !

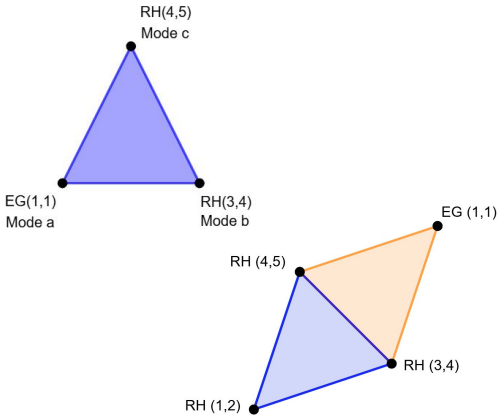




Final remarks

- Nonlinear spectral analysis of Rotating Shallow Water Equations on the Sphere in terms of Hough harmonics.
- Ongoing study of 3, 4, 5-wave systems: The kinetic energy fields of Rossby modes can be significantly altered by a gravity mode.
- Larger periods of energy transfer can be generated, exceeding 50 days. Fast -> Slow.

$$H(\lambda, \phi) = \Theta(\phi) e^{i\lambda m}$$



Gravity waves through nonlinear interactions change the slow dynamics of Rossby Haurwitz waves significantly in some cases.

Final remarks

Gravity waves through nonlinear interactions change the slow dynamics of Rossby Haurwitz waves significantly in some cases.

Outlook:

- Impacts of fast -> slow mode energy transfers in full shallow water system (PDE). How prevalent are they?

- How to make numerical models capture well these slow frequencies coming from fast frequencies?

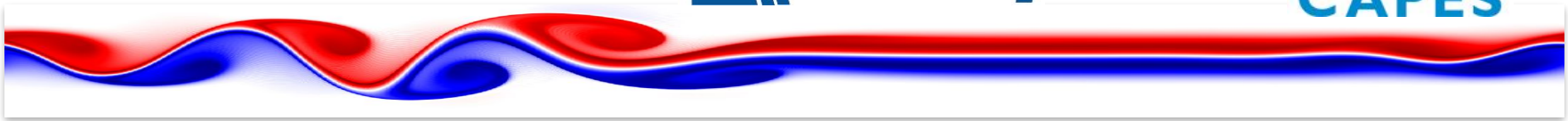
- Interpreting low frequency physical data through the eyes of nonlinear interactions: can they explain relevant climate phenomena?

Thanks!

Acknowledgements:

- Many collaborators, postdocs, students!
- FAPESP Jovem Pesquisador I e II/BPE
- CNPq Universal/Produtividade
- CAPES Auxílios/Bolsas

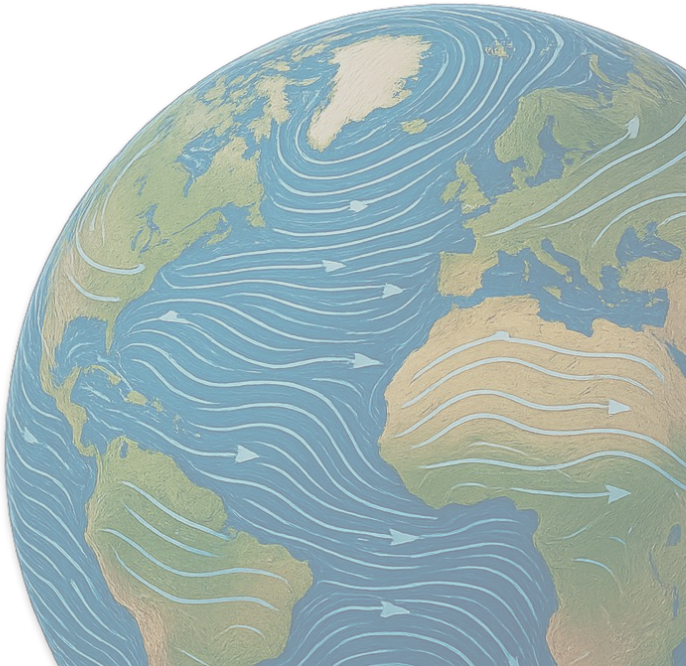
More info at: http://pedrosp.ime.usp.br



OBRIGADO !

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So far...



IME

FAPESP

COMMUN. IN PARTIAL DIFFERENTIAL EQUATIONS, 21(3&4), 619-658 (1996)

AVERAGING OVER FAST GRAVITY WAVES FOR GEOPHYSICAL FLOWS WITH ARBITRARY POTENTIAL VORTICITY

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Abstract

Here a mathematically rigorous framework is developed for deriving new reduced simplified dynamical equations for geophysical flows with arbitrary potential vorticity interacting with fast gravity waves. The examples include the rotating Boussinesq and rotating shallow water equations in the quasi-geostrophic limit with vanishing Rossby number. For the spatial periodic case the theory implies that the quasi-geostrophic equations are valid limiting equations in the weak topology for arbitrary initial data. Furthermore, simplified reduced equations are developed for the fashion in which the vortical waves influence the gravity waves through averaging over specific gravity wave/vortical

P. Embid and A.Majda (1996)

- f - plane
- Averaging principle

The vortical (slow) modes move independently of the gravity modes, and in average, the solutions are in the quasi-geostrophic regime

Tellus (2006), 58A, 263–279
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TELLUS

Dynamics of resonantly interacting equatorial waves

By CARLOS F. M. RAUPP* and PEDRO L. SILVA DIAS, *Department of Atmospheric Sciences, Institute of Astronomy, Geophysics and Atmospheric Sciences, University of São Paulo, São Paulo/SP, Brazil*

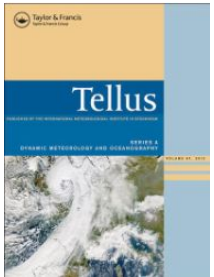
(Manuscript received 24 August 2004; in final form 30 April 2005)

ABSTRACT

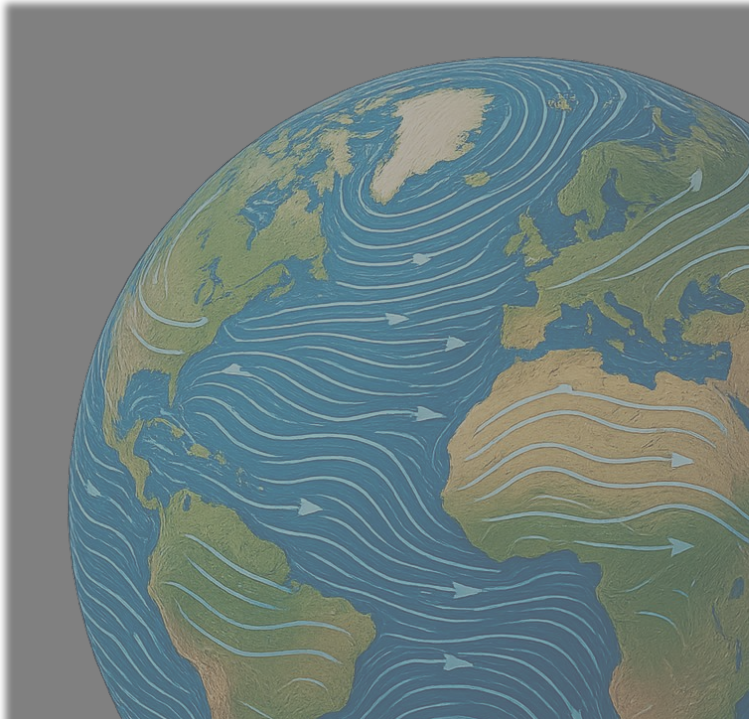
In this paper we explore some dynamical features on the non-linear interactions among equatorial waves. The shallow-water equation model with the equatorial β-plane approximation is used for this purpose. The Galerkin method is applied to the governing equations with the basis functions given by the eigensolutions of the linear problem. From the phase space expansion of two particular integrals of motion of the system, quadratic to lowest order, some constraints are obtained which the coupling coefficients must satisfy in order to ensure the invariance of such integrals. From the numerical evaluation of the coupling coefficients, these constraints are used to determine the possible resonant triads among equatorial waves. Numerical integrations of the resonant three-wave problem show that the energy of the waves in a resonant triad evolves periodically in time, with the period and amplitude of the energy oscillations dependent on the magnitude of the initial amplitudes of the waves and the way in which the initial energy is distributed among the triad components. The high-frequency modes are found to be energetically more active than the low-frequency modes. The latter tend to act as 'catalytic' components in a resonant triad. Integrations of the problem of two resonant triads coupled by a single mode point out the importance of gravity waves in the intertriad energy exchanges, suggesting the significance of these modes in the redistribution of energy throughout the atmospheric motion spectrum. The results also show that the intertriad energy exchanges provided by the highest frequency mode of two triads occur in a longer time-scale than the intratriad interactions. Therefore, these results also suggest the importance of the high-frequency modes in the generation of the low-frequency variability (intraseasonal and even longer term) of the atmospheric flow.

C. Raupp and P.S. Dias (2005)

- Equatorial β -plane
- Rossby and gravity waves
- Redistribution of energy
- Rossby waves altering periods



So far, but on the sphere



IME

FAPESP

Physics of Fluids ARTICLE scitation.org/journal/phf

Precession resonance of Rossby wave triads and the generation of low-frequency atmospheric oscillations

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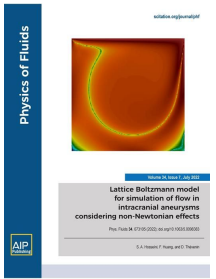
ABSTRACT

The dynamics of the Earth's atmosphere is characterized by a wide spectrum of oscillations, ranging from hourly to interdecadal and beyond. The low-frequency component of the atmospheric variability cannot be understood solely in terms of linear atmospheric waves that have shorter timescales. A newly proposed mechanism, the precession resonance mechanism, is a regime of highly efficient energy transfer in the spectral space in turbulent systems. Here, we investigate the role of the precession resonance, and the alignment of dynamical phases, in the generation of low-frequency oscillations and the redistribution of energy/enstrophy in the spectral space using the barotropic vorticity equation. First, the mechanism and its ability to generate low-frequency oscillations are demonstrated in low-order models consisting of four and five nonlinearly interacting Rossby–Haurwitz waves. The precession resonance onset is also investigated in the full barotropic vorticity equation, and the results are in agreement with the reduced models. Efficiency peaks in the energy/enstrophy transfer also correspond to regimes of strong excitation of low-frequency oscillations. The results suggest that the organization of the dynamical phases plays a key role in the redistribution of energy in the spectral space, as well as the generation of low frequencies in the barotropic vorticity equation.

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B. Raphaldini et al. (2022)

- Barotropic nondivergent
- Just Rossby-Haurwitz waves
- Precession resonance mechanism



stronger energy transfer

J. Adv. Model. Earth Syst., **Vol. 1,** Art. #2, 17 pp.

Shallow Water Quasi-Geostrophic Theory on the Sphere

Wayne H. Schubert, Richard K. Taft, and Levi G. Silvers

Department of Atmospheric Science, Colorado State University, Fort Collins, Colorado, USA

Manuscript submitted 22 July 2008; in final form 22 October 2008

Quasi-geostrophic theory forms the basis for much of our understanding of mid-latitude atmospheric dynamics. The theory is typically presented in either its f-plane form or its β-plane form. However, for many applications, including diagnostic use in global climate modeling, a fully spherical version would be most useful. Such a global theory does in fact exist and has for many years, but few in the scientific community seem to have ever been aware of it. In the context of shallow water dynamics, it is shown that the spherical version of quasi-geostrophic theory is easily derived (re-derived) based on a partitioning of the flow between nondivergent and irrotational components, as opposed to a partitioning between geostrophic and ageostrophic components. In this way, the invertibility principle is expressed as a relation between the streamfunction and the potential vorticity, rather than between the geopotential and the potential vorticity. This global theory is then extended by showing that the invertibility principle can be solved analytically using spheroidal harmonic transforms, an advancement that greatly improves the usefulness of this "forgotten" theory. When the governing equation for the time evolution of the potential vorticity is linearized about a state of rest, a simple Rossby-Haurwitz wave dispersion relation is derived and examined. These waves have a horizontal structure described by spheroidal harmonics, and the Rossby-Haurwitz wave frequencies are given in terms of the eigenvalues of the spheroidal harmonic operator. Except for sectoral harmonics with low zonal wavenumber, the quasi-geostrophic Rossby-Haurwitz frequencies agree very well with those calculated from the primitive equations. One of the many possible applications of spherical quasi-geostrophic theory is to the study of quasi-geostrophic turbulence on the sphere. In this context, the theory is used to derive an anisotropic Rhines barrier in three-dimensional wavenumber space.

A Balanced Approximation of the One-Layer Shallow-Water Equations on a Sphere

W. T. M. VERKLEY

Royal Netherlands Meteorological Institute (KNMI), De Bilt, Netherlands

(Manuscript received 16 May 2008, in final form 11 November 2008)

ABSTRACT

A global version of the equivalent barotropic vorticity equation is derived for the one-layer shallow-water equations on a sphere. The equation has the same form as the corresponding beta plane version, but with one important difference: the stretching (Cressman) term in the expression of the potential vorticity retains its full dependence on f^2, where f is the Coriolis parameter. As a check of the resulting system, the dynamics of linear Rossby waves are considered. It is shown that these waves are rather accurate approximations of the westward-propagating waves of the second class of the original shallow-water equations. It is also concluded that for Rossby waves with short meridional wavelengths the factor f^2 in the stretching term can be replaced by the constant value f_0^2, where f_0 is the Coriolis parameter at $\pm 45°$ latitude.

W. Schubert et al. / W. Verkley (2009)

- Quasi-geostrophic model
- Generalization of the barotropic nondivergent model
- Good approximation for the slow dynamics of the Shallow Water model

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NONLINEAR WAVE PROPAGATION ON A SPHERE: INTERACTION BETWEEN ROSSBY WAVES AND GRAVITY WAVES; STABILITY OF THE ROSSBY WAVES

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(*Received October 4 1993; in final form 13 April 1994*)

An analytical spectral model of the barotropic divergent equations on a sphere is developed using the potential-stream function formulation and the normal modes as basic functions. Explicit expressions of the coefficients of nonlinear interaction are obtained in the asymptotic case of a slowly rotating sphere, i.e. when the normal modes can be expressed as single spherical harmonics.

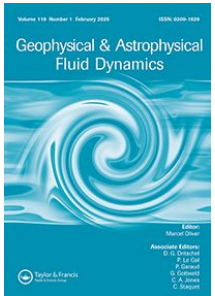
Highly truncated versions of the model are used to illustrate some consequences of the interaction between the Rossby waves and the surface gravity waves. So, it is shown how the gravity waves can exchange energy through interaction with a Rossby wave. A particular interaction between a Rossby wave and a gravity wave of zonal wavenumbers m and $2m$, respectively, is discussed in greater detail and is shown to lead to a frequency shift of the Rossby wave.

The stability of the Rossby waves to perturbations involving one or two gravity waves is examined and it is shown that the corresponding critical amplitudes are unrealistically large. The decay of the Rossby waves will probably be governed by the interactions inside Rossby waves triads.

J. Vannest and F. Vial (1994)

- Slowly rotating sphere
- Rossby-Haurwitz ⟶ rotational

  Gravity waves ⟶ divergent
- Analytical solutions



Normal modes – hough

- Spherical vector harmonics

$$H_n^m(\lambda,\phi) = \sum_{|n|\geq m} \left(iA_n^m y_{n,1}^m + B_n^m y_{n,2}^m - C_n^m y_{n,3}^m\right)$$

- Infinite system - pentadiagonal

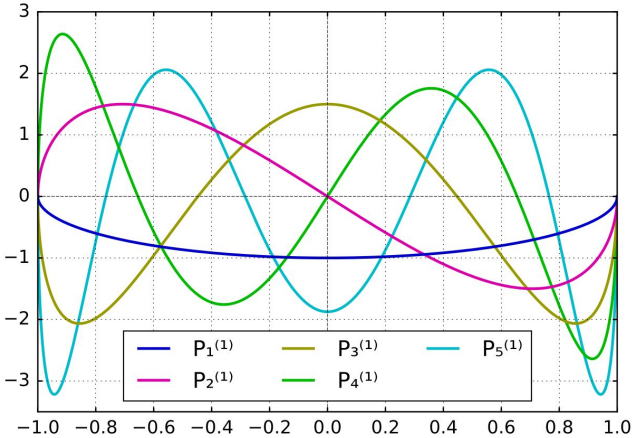
$$P_m X = \omega X$$

$$X = (C_m^m, A_m^m, B_{m+1}^m, C_{m+2}^m, A_{m+2}^m, \ldots)^T$$

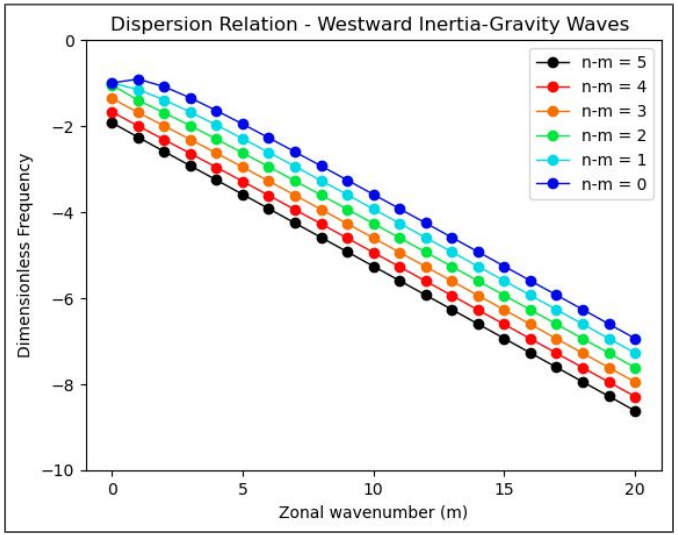
$$y_{n,1}^m = \begin{pmatrix} \frac{im}{\cos\phi} P_n^m \\ \frac{d}{d\phi} P_n^m \\ 0 \end{pmatrix} \frac{e^{i\lambda m}}{\sqrt{n(n+1)}}$$

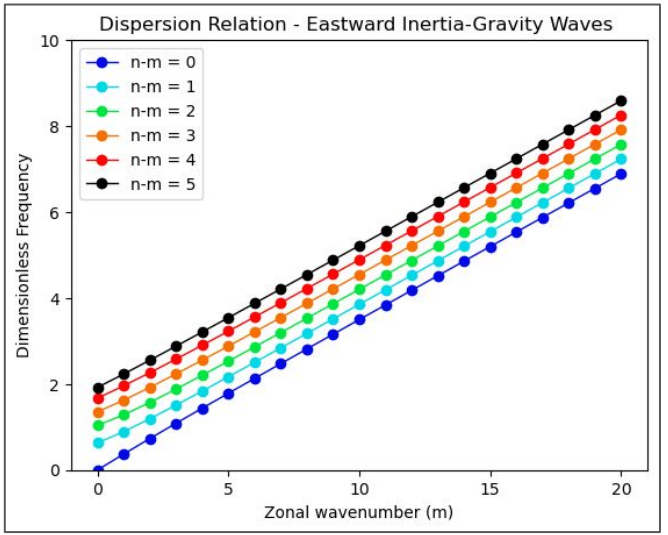
$$y_{n,2}^m = \begin{pmatrix} -\frac{d}{d\phi} P_n^m \\ \frac{im}{\cos\phi} P_n^m \\ 0 \end{pmatrix} \frac{e^{i\lambda m}}{\sqrt{n(n+1)}}$$

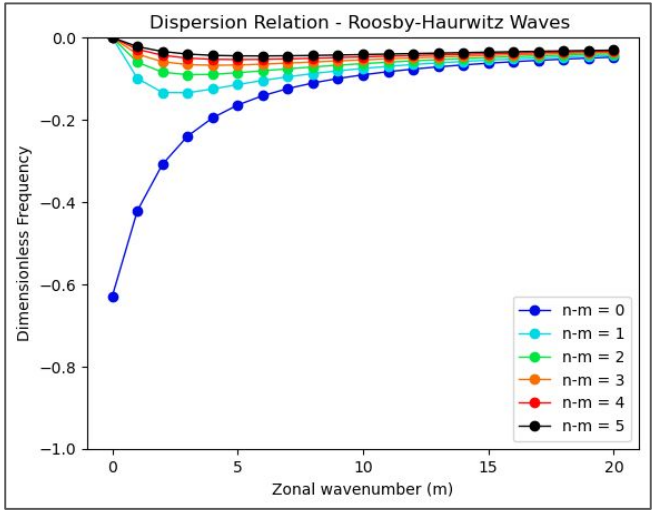
$$y_{n,3}^m = \begin{pmatrix} 0 \\ 0 \\ P_n^m \end{pmatrix} e^{i\lambda m}$$



P. Swarztrauber and A. Kasahara (1985).


Dispersion Relation - Westward Inertia-Gravity Waves
Dimensionless Frequency
Zonal wavenumber (m)
n-m = 5
n-m = 4
n-m = 3
n-m = 2
n-m = 1
n-m = 0


Dispersion Relation - Eastward Inertia-Gravity Waves
Dimensionless Frequency
Zonal wavenumber (m)
n-m = 0
n-m = 1
n-m = 2
n-m = 3
n-m = 4
n-m = 5


Dispersion Relation - Roosby-Haurwitz Waves
Dimensionless Frequency
Zonal wavenumber (m)
n-m = 0
n-m = 1
n-m = 2
n-m = 3
n-m = 4
n-m = 5

$$h_0 = 10km$$

Hough harmonics

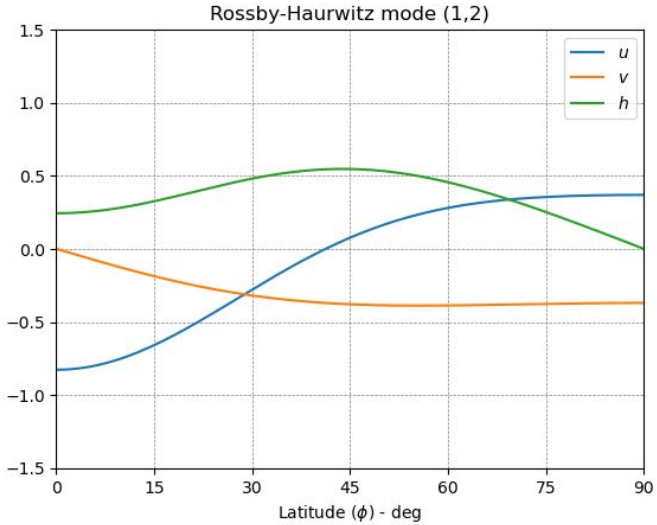
$\alpha = 1$: eastward inertio-gravity (EIG)
$\alpha = 2$: westward intertio-gravity (WIG)
$\alpha = 3$: Rossby-Haurwitz (RH)

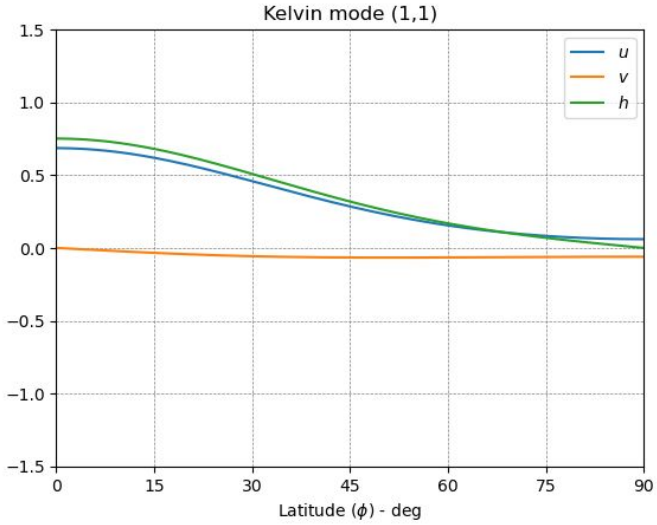
$$H^m_{n,\alpha}(\lambda,\phi) = \Theta^m_{n,\alpha}(\lambda,\phi)e^{i\lambda m}$$

$$\Theta^m_{l,\alpha}(\phi) = \begin{pmatrix} u^m_{l,\alpha}(\phi) \\ i\ v^m_{l,\alpha}(\phi) \\ h^m_{l,\alpha}(\phi) \end{pmatrix}$$

Hough harmonic

Hough function





P. Swarztrauber and A. Kasahara (1985).

Energy

- Kinetic Energy

$$E^{(2)}(t) = \frac{1}{2}\langle \vec{u}, \vec{u} \rangle = |A_a|^2 + |A_b|^2 + |A_c|^2$$

Total energy

$$E_T(t) = E^{(2)}(t) + E^{(3)}(t)$$

is conserved

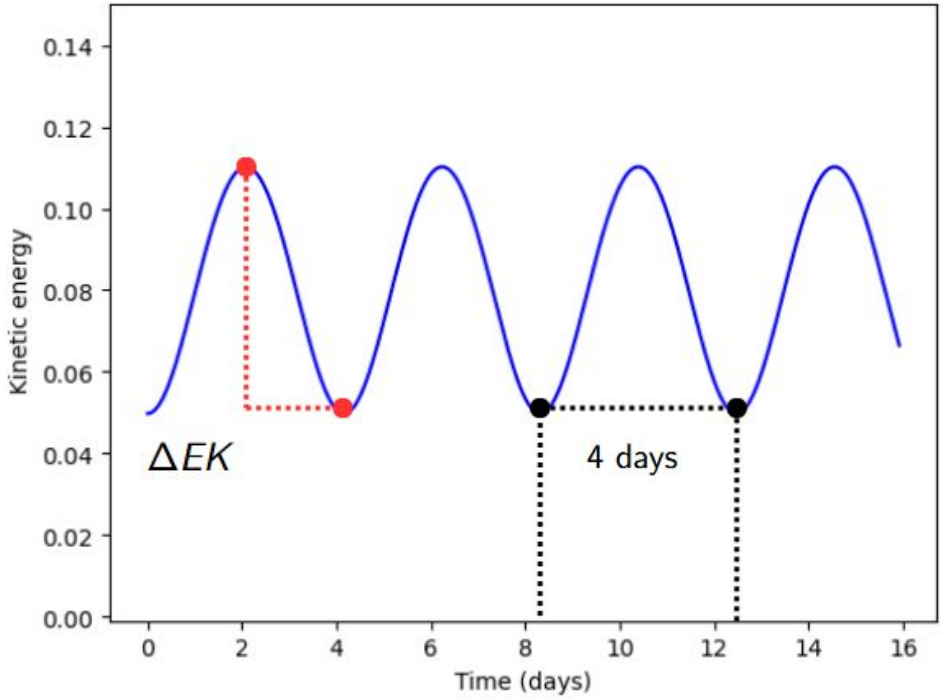
- Cubic Energy

$$\begin{aligned} E^{(3)}(t) &= \frac{1}{4\pi}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} h(|u|^2 + |v|^2)\cos\phi \; d\phi \; d\lambda \\ & \quad 2\mathbb{R}e(A_a{}^* A_b{}^* A_c)\int_{-\pi/2}^{\pi/2} [h_a(u_b u_c + v_b v_c) + h_b(u_a u_c + v_a v_c) \\ &+ \quad h_c(u_a u_b - v_a v_b)\,]\cos\phi \; d\phi \end{aligned}$$

T. Schepherd. *Symmetries, conservation laws, and hamiltonian structure in geophysical fluid dynamics.* Advances in Geophysics, 1990.

Efficiency

$$\mathscr{E} = \frac{\max_t |A_a(t)|^2 - \min_t |A_a(t)|^2}{E_{total}(0)}$$

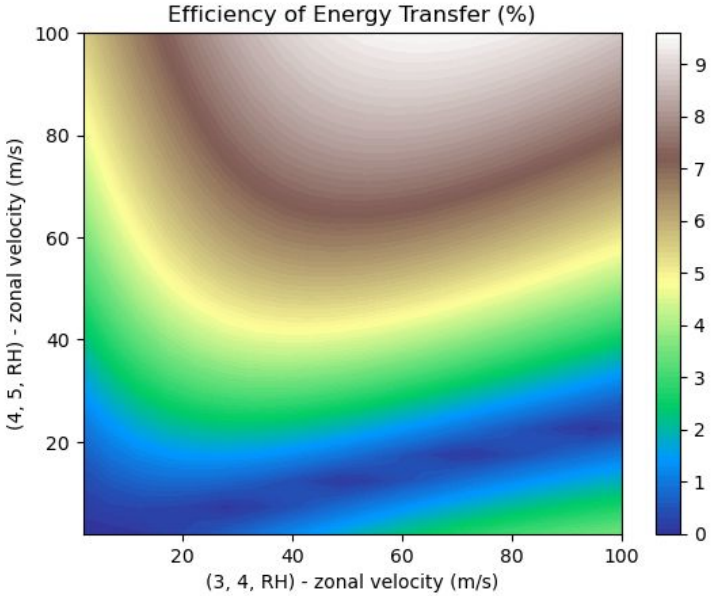


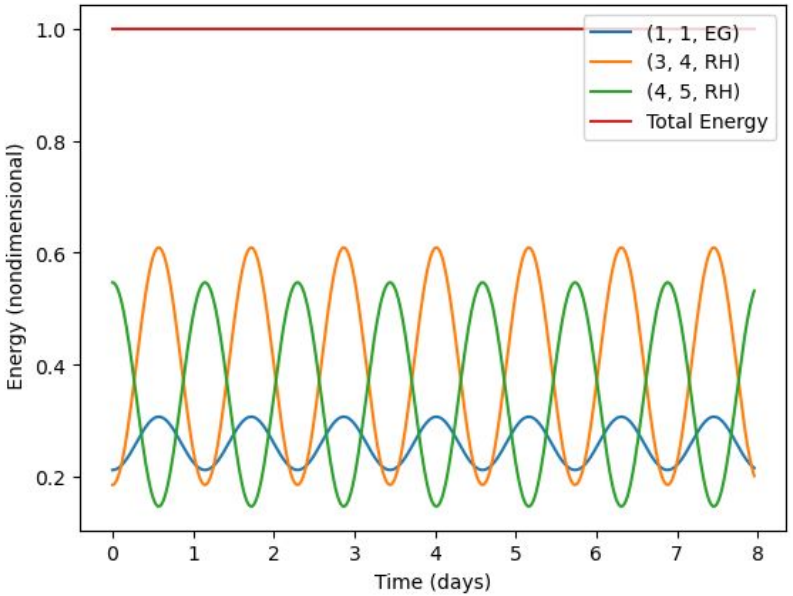
Higher efficiency → most of the total energy → stronger interactions

M. D. Bustamante, B. Quinn, and D. Lucas. *Robust energy transfer mechanism via precession resonance in nonlinear turbulent wave systems.* Phys. Rev. Lett., 2014.

RH ⟶ EIG

		Coup. coeff.	Freq.	Zonal (m/s)	Amp.
Mode a	1,1,EG	-0.13256	0.3691	80	0.1905
Mode b	3,4,RH	-0.59112	-0.1343	63.89	0.1782
Mode c	4,5,RH	-0.55801	-0.1252	100	0.3061
Mismatch	0.3606				
Efficiency	9.503 %				





More on three waves

- Pump mode ⟶ receives/conceals energy
- Periodic solutions
- Low efficiency for RH modes (in some cases)

	Case I		Case II
$\alpha^a_{b,c}$	+	-	-
$\alpha^b_{a,c}$	+	-	+
$\alpha^c_{a,b}$	+	-	-
Pump mode	mode c		mode a

	Case III	Case IV	
$\alpha^a_{b,c}$	-	+	-
$\alpha^b_{a,c}$	+	+	-
$\alpha^c_{a,b}$	+	-	+
Pump mode	mode b	none	

C. Raupp and P. S. Dias. *Dynamics of resonantly interacting equatorial waves*. Tellus, 2006.
J. Harris. *The kinematics, dynamics and statistics of three-wave interactions in models of geophysical.* PhD thesis, University of Warwick, 2013.
E. Kartashova and M. Bustamante. *Resonance clustering in wave turbulent regimes: Integrable dynamics.* Communications in Computational Physics, 2011.

Four Waves

Bad news

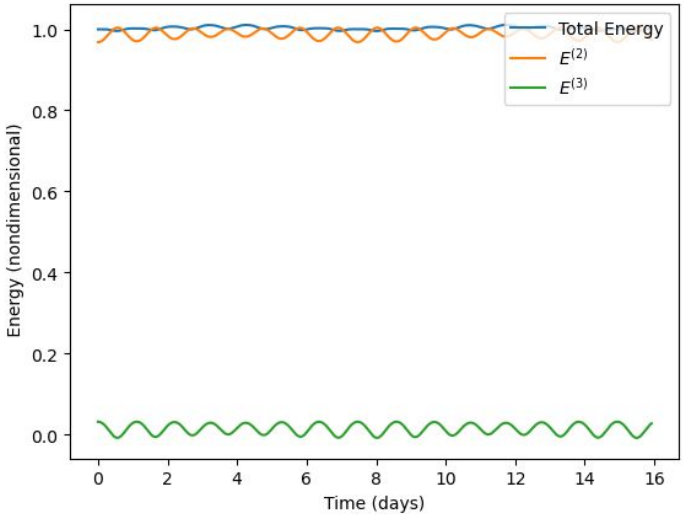
The system does not necessarily conserve energy

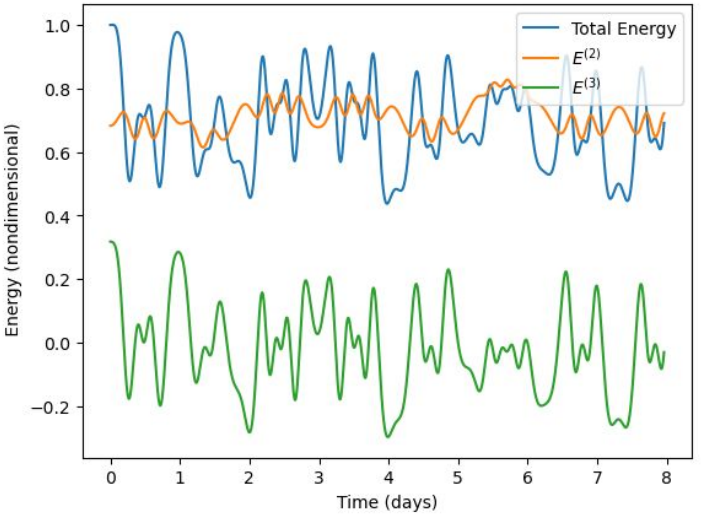
Good news

Variations are

$$O(A^4)$$

→ our case: less than 2%

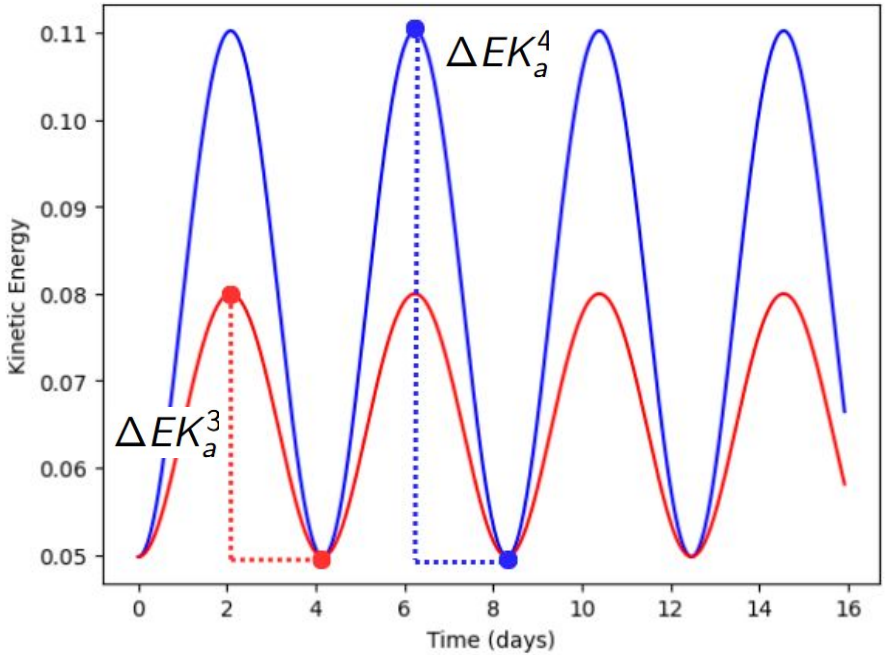




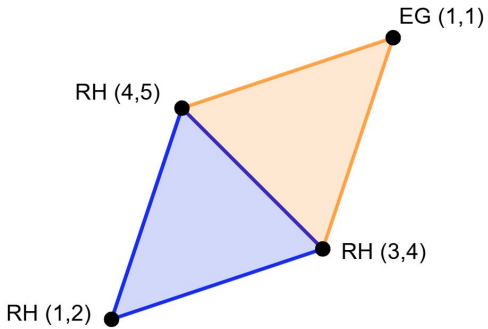
$\mathscr{P}$- measure

$$\mathscr{P}_a = \frac{\Delta EK_a^4(t) - \Delta EK_a^3(t)}{\Delta EK_a^3(t)}$$

Measures the percent increase of kinetic energy due to the presence of a gravity mode



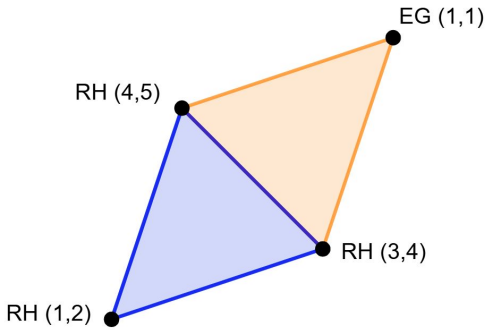
Example



		Freq.	Coeff. 1	Coeff. 2
Mode a	4,5,RH	-0.12522	0.3638	-0.5572
Mode b	3,4,RH	-0.13431	0.4434	-0.5902
Mode c	1,2,RH	-0.09934	-0.0284	-
Mode d	1,1,EG	0.36913	-	-0.1323

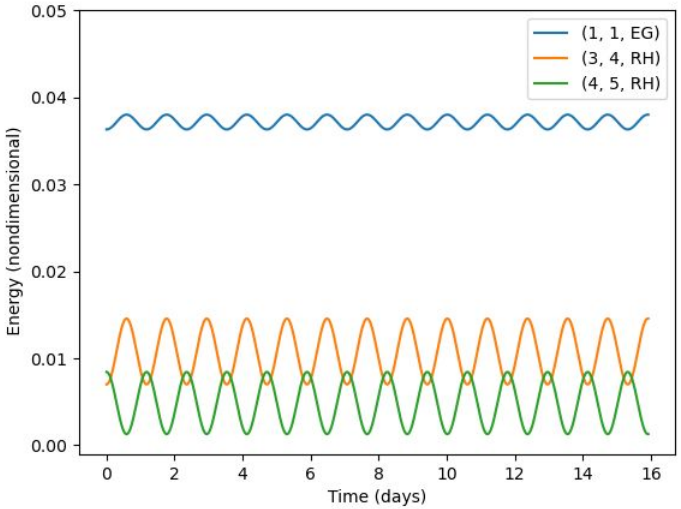
	Example 1		Example 2		Example 3	
Mode	Zonal	Amplitude	Zonal	Amplitude	Zonal	Amplitude
4,5,RH	30	0.0918137	30	0.0918137	30	0.122418
3,4,RH	30	0.0836648	30	0.0836648	30	0.111553
1,2,RH	30	0.0695306	30	0.0695306	30	0.0695306
1,1,EG	0	0	80	0.1905199	20	0.04763

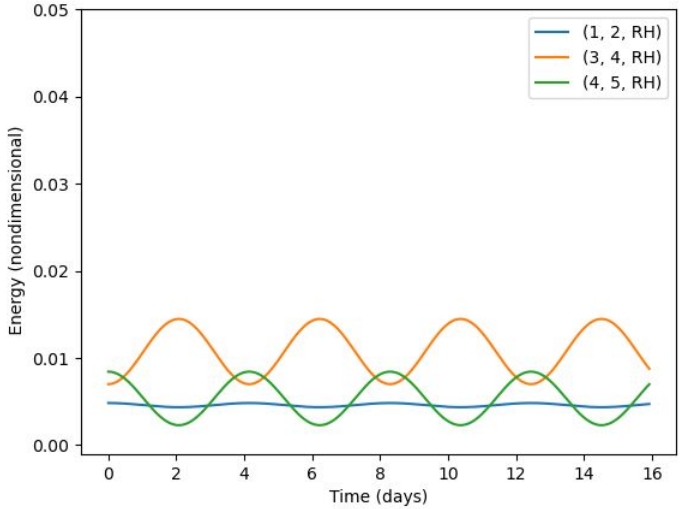
Example

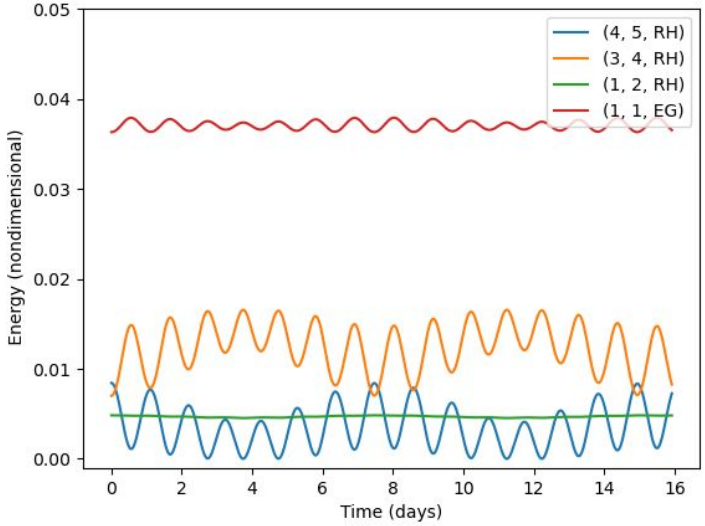


		Freq.	Coeff. 1	Coeff. 2
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Mode b	3,4,RH	-0.13431	0.4434	-0.5902
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Mode d	1,1,EG	0.36913	-	-0.1323

	Example 1		Example 2		Example 3	
Mode	Zonal	Amplitude	Zonal	Amplitude	Zonal	Amplitude
4,5,RH	30	0.0918137	30	0.0918137	30	0.122418
3,4,RH	30	0.0836648	30	0.0836648	30	0.111553
1,2,RH	30	0.0695306	30	0.0695306	30	0.0695306
1,1,EG	0	0	80	0.1905199	20	0.04763


(1, 1, EG)
(3, 4, RH)
(4, 5, RH)
Energy (nondimensional)
Time (days)
0.00
0.01
0.02
0.03
0.04
0.05
0
2
4
6
8
10
12
14
16

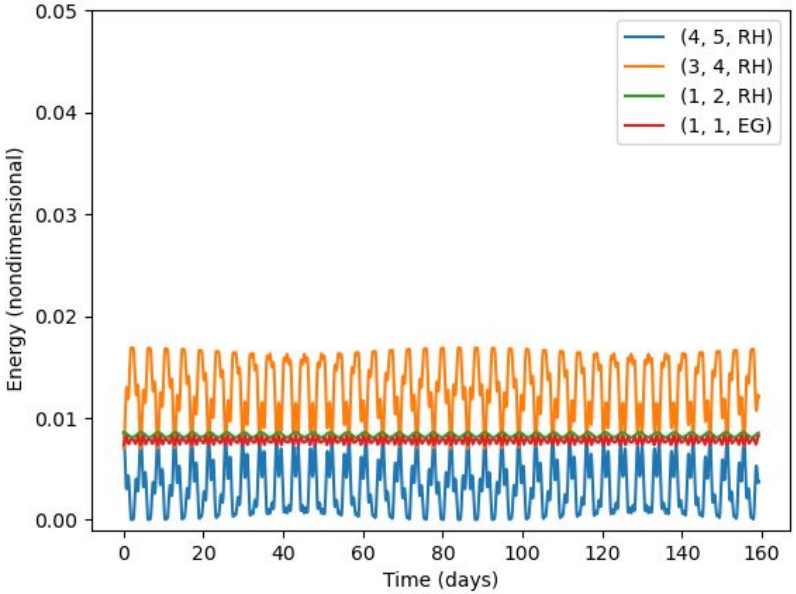

(1, 2, RH)
(3, 4, RH)
(4, 5, RH)
Energy (nondimensional)
Time (days)
0.00
0.01
0.02
0.03
0.04
0.05
0
2
4
6
8
10
12
14
16

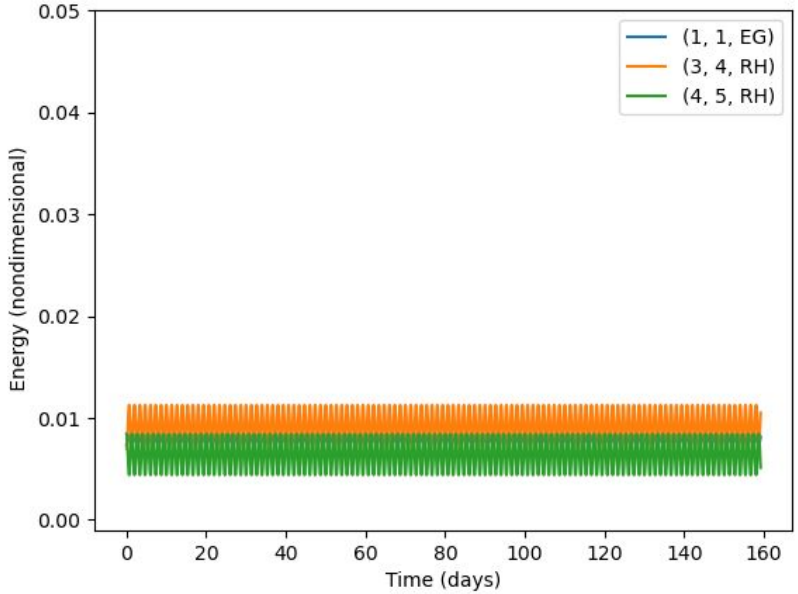

(4, 5, RH)
(3, 4, RH)
(1, 2, RH)
(1, 1, EG)
Energy (nondimensional)
Time (days)
0.00
0.01
0.02
0.03
0.04
0.05
0
2
4
6
8
10
12
14
16

$\mathscr{P}$ – measure

Mode	ΔEK_1^3	ΔEK_2^3	ΔEK^4	$\mathscr{P}(\%)$
4,5,RH	0.006136	0.007142	0.008426	37.3115
3,4,RH	0.007478	0.00757	0.009549	27.70768
1,2,RH	0.0004796	-	0.000307	-35.9831
1,1,EG	-	0.0016962	0.001604	-5.43584

- RH modes (4,5) and (3,4) receive more energy in the quartet
- EG mode (1,1) loses kinetic energy


(4, 5, RH)
(3, 4, RH)
(1, 2, RH)
(1, 1, EG)
Energy (nondimensional)
Time (days)
0.00
0.01
0.02
0.03
0.04
0.05
0
20
40
60
80
100
120
140
160


(1, 1, EG)
(3, 4, RH)
(4, 5, RH)
Energy (nondimensional)
Time (days)
0.00
0.01
0.02
0.03
0.04
0.05
0
20
40
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