Nonlinear wave interactions and the generation of low-frequency atmospheric oscillations

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PDEs on the Sphere - July 2023

Summary

1. Quick introduction to Rossby-Harwitz nonlinear wave interaction theory
2. Resonances in wave triads
3. The precession mechanism - Interacting Triads!
4. Results in truncated ODE model
5. Results in spherical PDE model

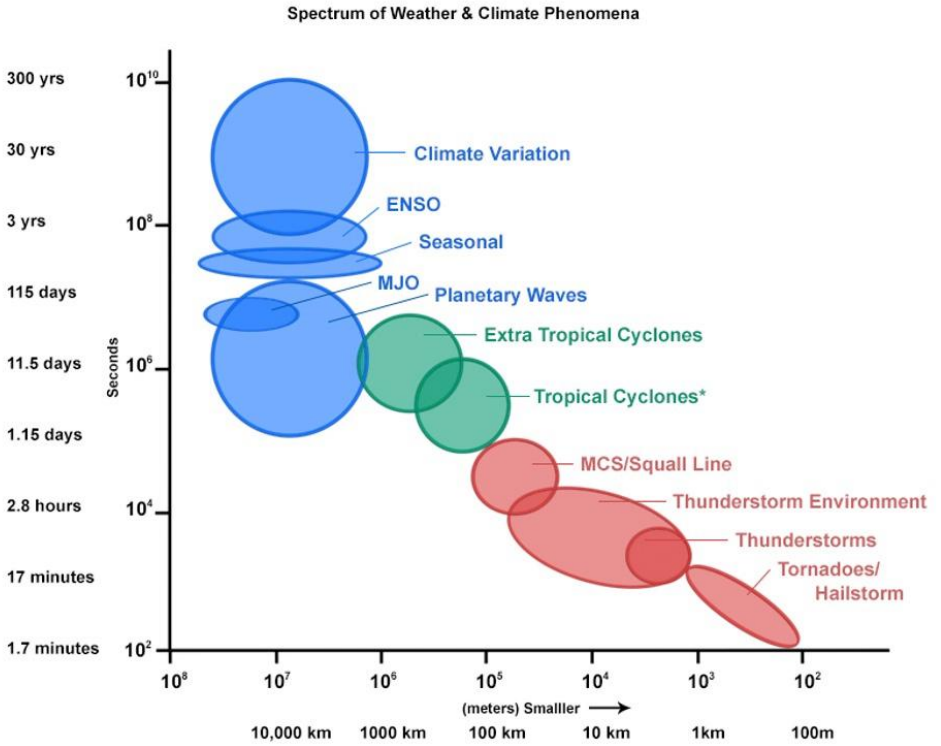
Why study this?

- Much is known about linear aspects of the atmosphere dynamics, but a lot is still to be learned from the nonlinear dynamics, that can potentially explain certain sub-seasonal/seasonal phenomena.

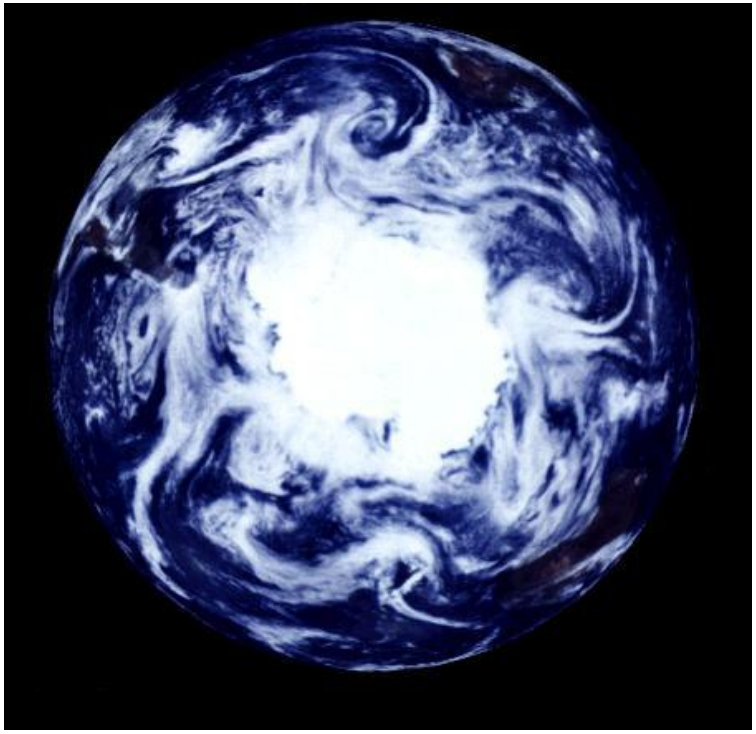
Why show this in PDEs on the Sphere Workshop?

- Proper treatment of nonlinear interactions pose a great challenge for time stepping methods and particularly for parallel-in-time methods

A sea of “noise” in large-scale motions



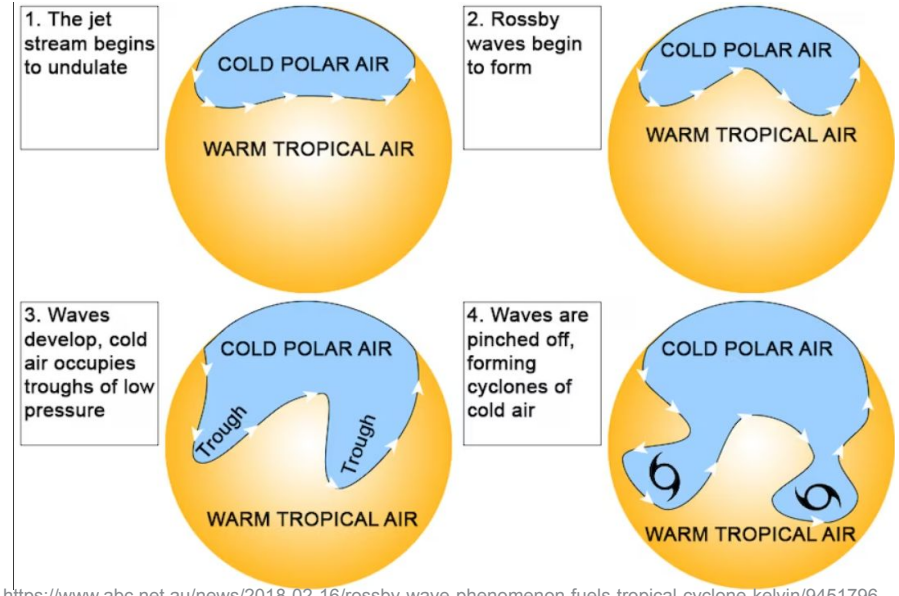
Scales of Atmospheric Motions (Australian Bureau of Meteorology)
https://cycwxchallenge.wordpress.com/2020/05/22/the-forecast-funnel/



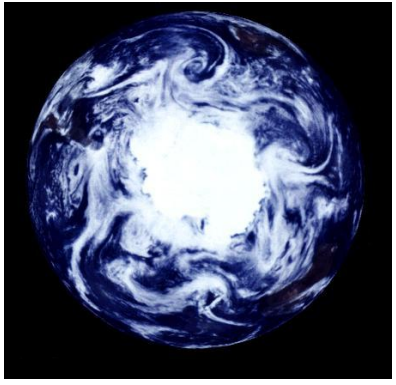
Mosaic of 24h images from Galileo Orbiter 1997
https://www.jpl.nasa.gov/images/pia00729-south-polar-projection-of-earth

A sea of “noise” in large-scale motions

Rossby-Harwitz Waves



https://www.abc.net.au/news/2018-02-16/rossby-wave-phenomenon-fuels-tropical-cyclone-kelvin/9451796



Mosaic of 24h images from Galileo Orbiter 1997
https://www.jpl.nasa.gov/images/pia00729-south-polar-projection-of-earth

Barotropic nondivergent model

- Rossby (1939) - Beta-plane
- Haurwitz (1940) - Sphere
- Charney et al (1950) - NWP

$$\frac{D\eta}{Dt} = 0$$

$$\eta = \zeta + f$$

Conservation of absolute vorticity

Barotropic non-divergent model

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot ((\zeta + f)\vec{v}) = 0.$$

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \phi} + J(\psi, \nabla^2 \psi) = 0,$$

$\vec{v}(\phi, \theta)$ - Horizontal velocity

$\zeta = (\nabla \times \vec{v}) \cdot \vec{r}$ - Relative vorticity

(ϕ, θ) - Lattitude, longitude

$$\zeta = -\nabla^2 \psi$$

$$\vec{v} = \nabla \times (\psi \vec{r})$$

Stream function ψ

$$J(h, g) = \frac{1}{a^2 \cos\theta}\left(\frac{\partial h}{\partial \theta}\frac{\partial g}{\partial \phi} - \frac{\partial h}{\partial \phi}\frac{\partial g}{\partial \theta}\right)$$

Conservation of total energy and total enstrophy

Rossby-Harwitz wave theory

$$\psi(\phi,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{n,m}(t) Y_n^m(\phi,\theta) e^{-i\omega_{n,m}t}$$

Y_n^m Spherical harmonics

Linear theory is well known and understood!

Rossby-Haurwitz waves linear dispersion relations:

$$\omega_{n,m} = -\frac{2\Omega m}{n(n+1)}$$

Nonlinear wave interaction theory goes back to Platzman (1960, 1962)

$$\frac{dA_{k_1}}{dt} = \sum_{k_2}\sum_{k_3} iC_{k_2k_3}^{k_1} A_{k_2} A_{k_3} e^{i\delta t}$$

$k_i = (n_i, m_i)$ arbitrary spherical harmonic

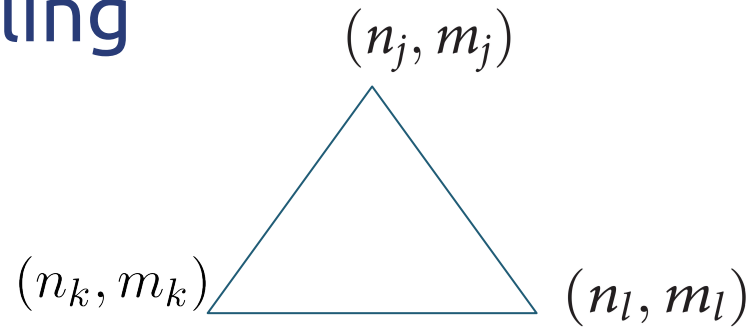
$C_{k_2k_3}^{k_1}$ nonlinear coupling coefficients

$\delta = \omega_{k_1} - \omega_{k_2} - \omega_{k_3}$ eigenfrequencies mismatch

$C_{k,l}^{j} \in \mathbb{R}$ Nonlinear interaction coefficients (known formula)

Nonlinear triad coupling

$$\frac{dA_j}{dt} = i \sum_k \sum_l C^j_{k,l} A_k A_l e^{-i\delta^j_{k,l} t}$$



$$C^j_{k,l} = \frac{(n_k(n_k+1) - n_l(n_l+1))}{a^2 n_j(n_j+1)} \int_{-1}^{1} P^{m_j}_{n_j}(z) \left(m_k P^{m_k}_{n_k}(z) \frac{dP^{m_l}_{n_l}(z)}{dz} - m_l P^{m_l}_{n_l}(z) \frac{dP^{m_k}_{n_k}(z)}{dz} \right) dz.$$

Coupling coefficients are only nonzero if they satisfy **Ellsaesser** conditions (right) and must satisfy certain relations related to **Fjørtoft's** theorem (bellow).

$$n_j(n_j+1)C^j_{k,l} - n_k(n_k+1)C^k_{j,l} - n_l(n_l+1)C^l_{j,k} = 0,$$
$$\left[n_j(n_j+1)\right]^2 C^j_{k,l} - \left[n_k(n_k+1)\right]^2 C^k_{j,l} - \left[n_l(n_l+1)\right]^2 C^l_{j,k} = 0.$$

$$m_k + m_l = m_j,$$
$$(m_j)^2 + (m_l)^2 \neq 0,$$
$$n_k n_j n_l \neq 0,$$
$$n_k + n_j + n_l \quad \text{is} \quad \text{odd},$$
$$(n_k^2 - |m_k|^2) + (n_l^2 - |m_l|^2) > 0,$$
$$|n_k - n_l| < n_j < n_k + n_l,$$
$$(m_k, n_k) \neq (m_j, n_j), (m_l, n_l) \neq (m_j, n_j)$$

Resonances and Pump mode

Resonance = the relative linear phase of the wave triplet not to change in time

$$\delta = \omega_{k_1} - \omega_{k_2} - \omega_{k_3} = 0.$$

Associated with the leading-order nonlinear effects for a weakly nonlinear regime of any dispersive wave system having quadratic nonlinearities.

Pump mode = mode with intermediate total wavenumber

$$|C^{j}_{k,l}| > |C^{k(l)}_{j,l(k)}| \Rightarrow n_{k(l)}(n_{k(l)}+1) < n_j(n_j+1) < n_{l(k)}(n_{l(k)}+1).$$

⇨ Relevant to excite mechanisms in later examples

Will always receive energy from or supply energy to the remaining triad components. For resonate triads, it has the highest absolute eigenfrequency.

Triad example and pump instability

$$\frac{dA_1}{dt} = iC^1_{2,3}A^*_2A_3e^{i\delta t},$$
$$\frac{dA_2}{dt} = iC^2_{3,1}A^*_1A_3e^{i\delta t},$$
$$\frac{dA_3}{dt} = iC^3_{1,2}A_1A_2e^{-i\delta t},$$

Assume A_3 is the pump mode, then…

Pump instability is triggered if

$$C^2_{3,1}C^1_{2,3} > 0 \qquad |A_3(0)| > \frac{|\delta|}{2\sqrt{C^2_{3,1}C^1_{2,3}}}.$$

In resonant and nearly resonant triad regimes: emergence of wave instability in typical weakly nonlinear regime.

Triad oscillation frequencies

Change of variables:

$$A_j(t) = i\,B_j(t)\mathrm{e}^{i\varphi_j(t)}$$

$$\frac{dB_1}{dt} = C^1_{2,3}B_2B_3\cos(\Phi),$$

$$\frac{dB_2}{dt} = C^2_{3,1}B_1B_3\cos(\Phi),$$

$$\frac{dB_3}{dt} = -C^3_{1,2}B_1B_2\cos(\Phi),$$

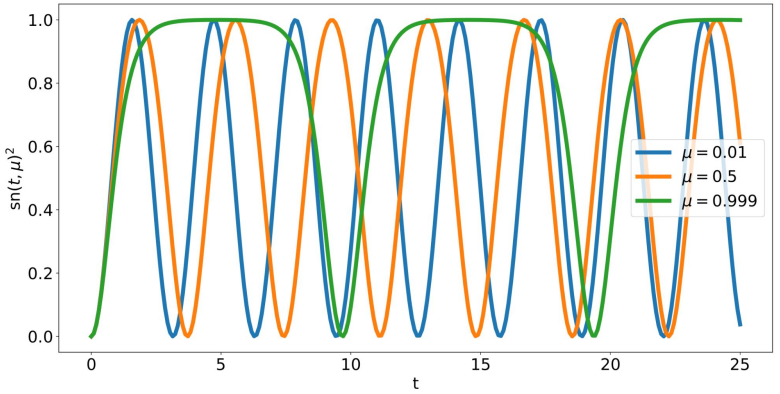
$$\frac{d\Phi}{dt} = -\delta - B_1B_2B_3\left(\frac{C^1_{2,3}}{B_1^2} + \frac{C^2_{3,1}}{B_2^2} - \frac{C^3_{1,2}}{B_3^2}\right)\sin(\Phi)$$

Dynamical phase

$$\Phi(t) = \varphi_1(t) + \varphi_2(t) - \varphi_3(t) - \delta\,t$$

The Isolated Triad System is integrable!

Jacobi elliptic functions ⇨ Solutions periodic in amplitudes



- Solutions for resonant case ($\delta = 0$) are periodic in time (period T):

$$T = \frac{3^{1/4}\sqrt{2}\,K(\mu)}{(1-\rho+\rho^2)^{1/4}\sqrt{\cos(\nu/3-\pi/6)}\sqrt{I_{13}}}.$$

Resonant Oscillation (Fourier) frequency

$$\Gamma = 2\pi/T$$ ⇨

Freq. proportional to initial amplitudes!

Triad oscillation frequencies

Change of variables:

$$A_j(t) = i\,B_j(t)\mathrm{e}^{i\varphi_j(t)}$$

$$\frac{dB_1}{dt} = C^1_{2,3}B_2B_3\cos(\Phi),$$

$$\frac{dB_2}{dt} = C^2_{3,1}B_1B_3\cos(\Phi),$$

$$\frac{dB_3}{dt} = -C^3_{1,2}B_1B_2\cos(\Phi),$$

$$\frac{d\Phi}{dt} = -\delta - B_1B_2B_3\left(\frac{C^1_{2,3}}{B_1^2} + \frac{C^2_{3,1}}{B_2^2} - \frac{C^3_{1,2}}{B_3^2}\right)\sin(\Phi)$$

Dynamical phase

$$\Phi(t) = \varphi_1(t) + \varphi_2(t) - \varphi_3(t) - \delta\,t$$

The Isolated Triad System is integrable!

- General solutions are periodic in amplitude
- Solutions for resonant case ($\delta = 0$) are fully periodic in time (period T):

Resonant Oscillation (Fourier) frequency

$$\Gamma = 2\pi/T$$

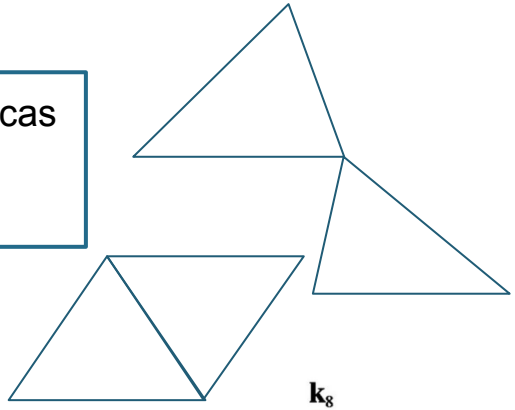
- In **non or near resonant** regimes the dynamical phase has additionally phase slips that occur periodically with period T, leading to "drift" or "**precession**".

Precession frequency:

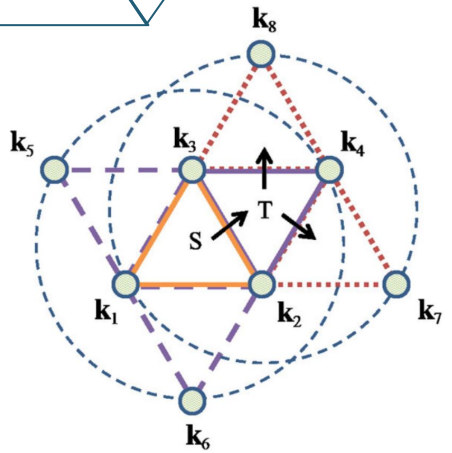
$$\Omega = \left\langle \frac{d\Phi}{dt} \right\rangle = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \left(\frac{d\Phi}{dt}\right)dt = \lim_{\tau\to\infty}\frac{\Phi(\tau)-\Phi(0)}{\tau}$$

Precession resonance between triads

The **precession** mechanism has been proposed by Bustamante, Quinn, and Lucas (2014) as a highly efficient **energy transfer** mechanism that takes place when **multiple wave triads are connected**. (For general nonlinear turbulence…)



In short: bursts of strong energy transfer are interchanged with periods of weak interactions; this interchange between **strong** and weak **energy** fluxes was found to be associated with **synchronized** and unsynchronized regimes, respectively, between amplitudes and dynamical phases of the wave triplets.



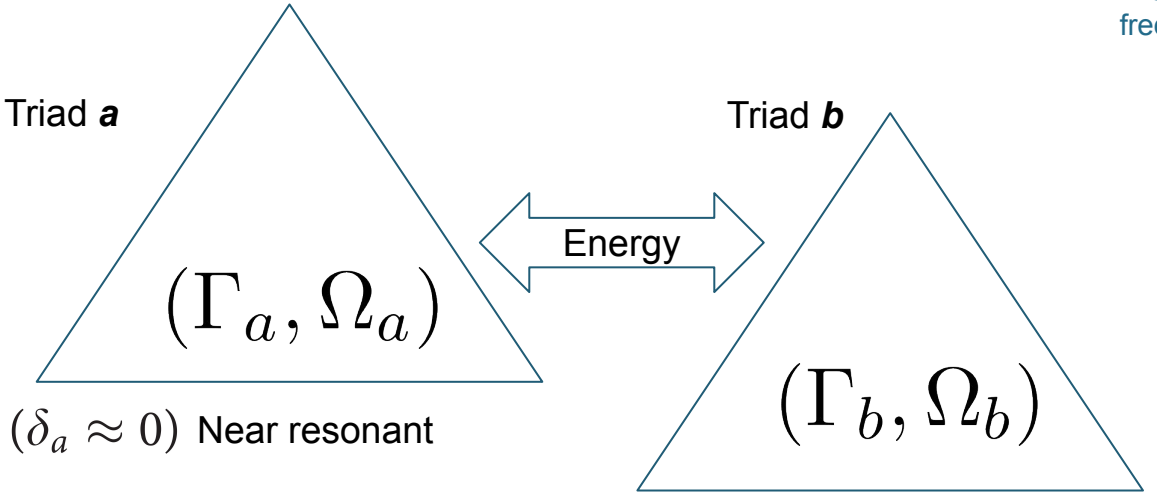
Of relevance: inter-triad energy transfers associated with the precession resonance occur in a **longer timescale than the intra-triad energy exchanges**, thus yielding longer-period modulations on the energy exchanges within a resonant triad. This aspect of the precession resonance should also be useful in the atmospheric dynamics context.

Precession resonance between triads

Systems with more than 1 triad are not integrable!

Harmonic Oscillation frequency $\Gamma = 2\pi/T$

Precession (drift) frequency $\Omega = \left\langle \frac{d\Phi}{dt} \right\rangle$



$(\delta_a \approx 0)$ Near resonant

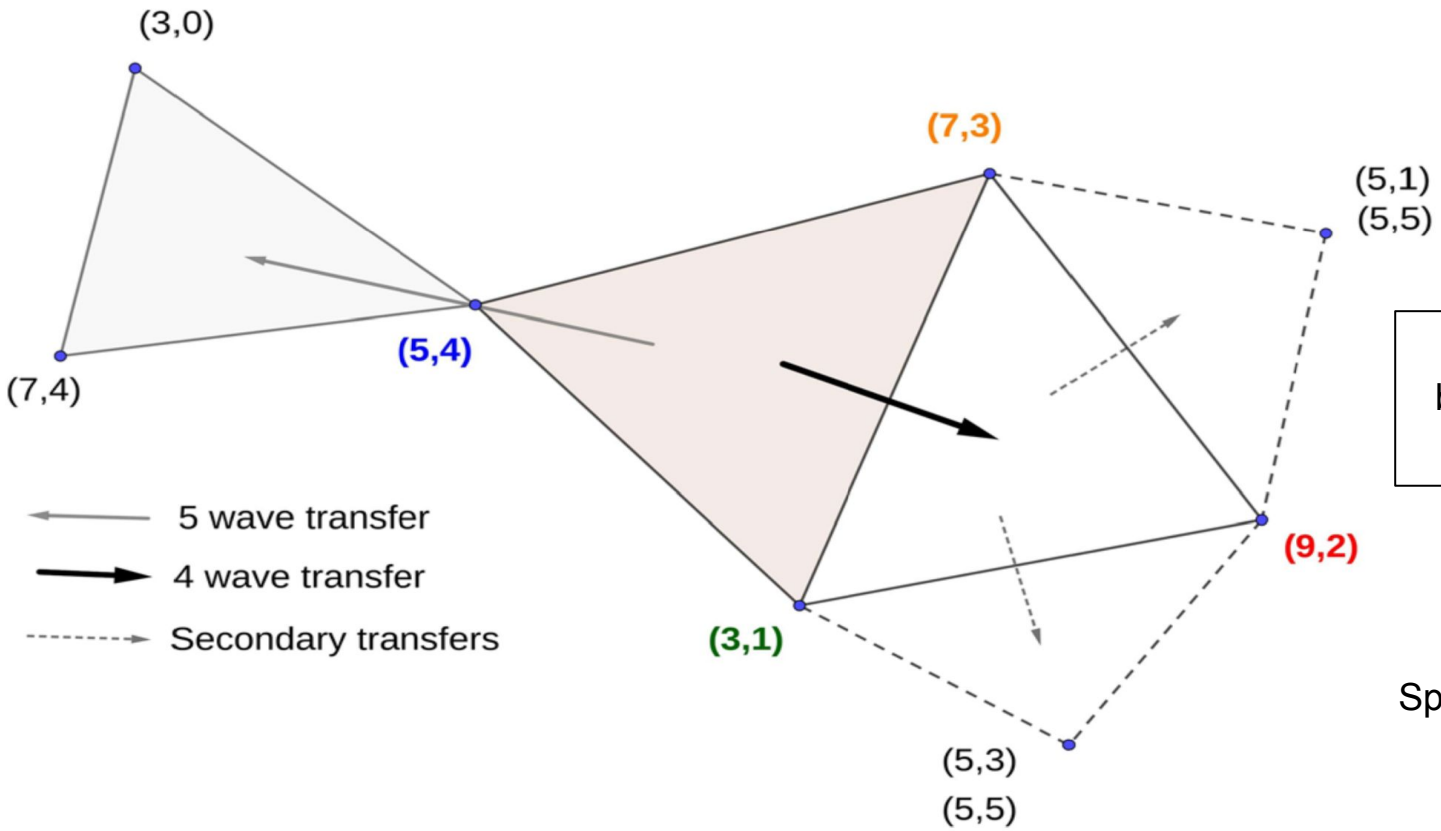
Precession resonance between triads a and b occur when the **precession frequency of triad b** matches one of the Fourier **harmonic frequencies** of the amplitude modulation of **triad a**:

$$\Omega_b \approx p \times \Gamma_a$$

(p integer)

How to test this? The period (T), and therefore the frequency (Γ) scales with initial condition amplitude of modes, so we can vary it to find peaks of energy transfers! α

Truncated ODE Examples

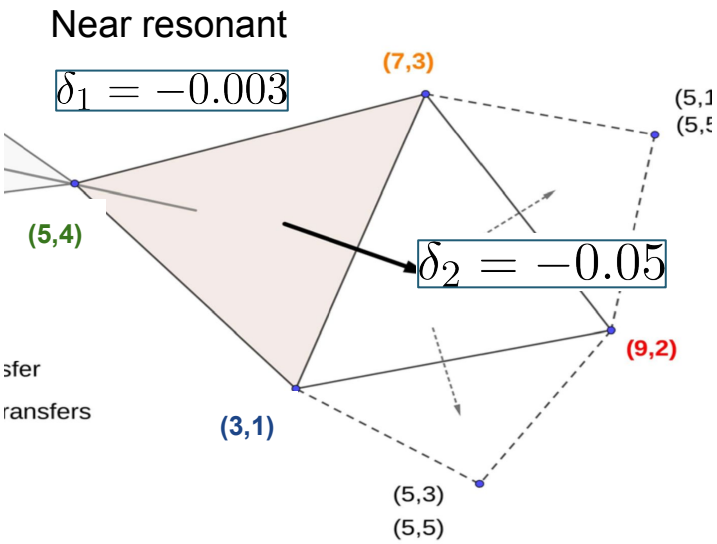


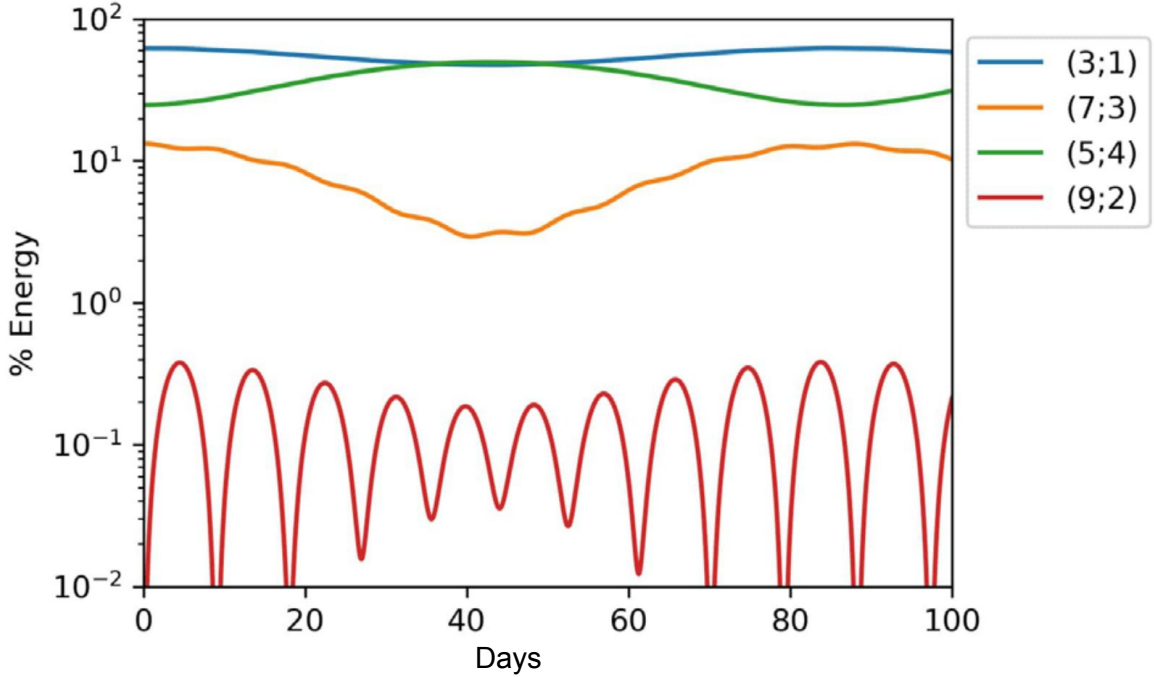
Expected transfers based on **Ellsaesser** conditions

(n, m)

Spherical Harmonic modes

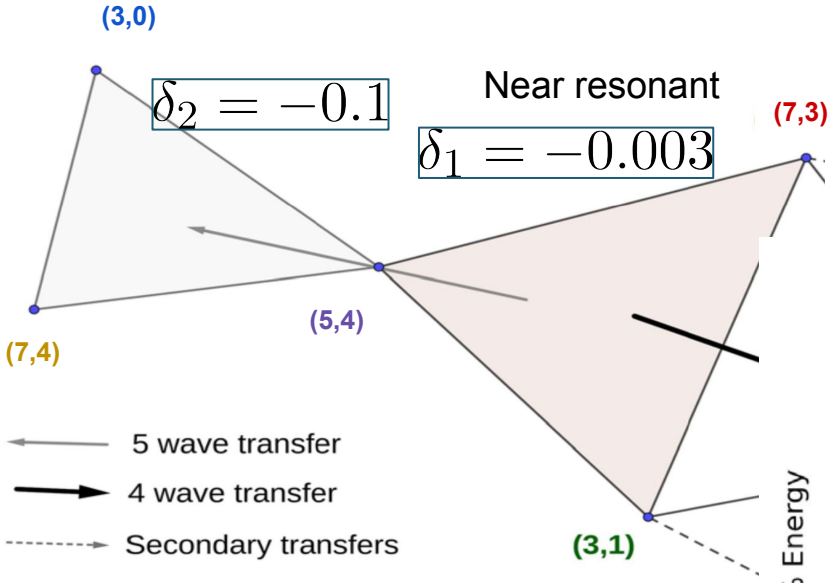
4 wave example (truncated ODE)

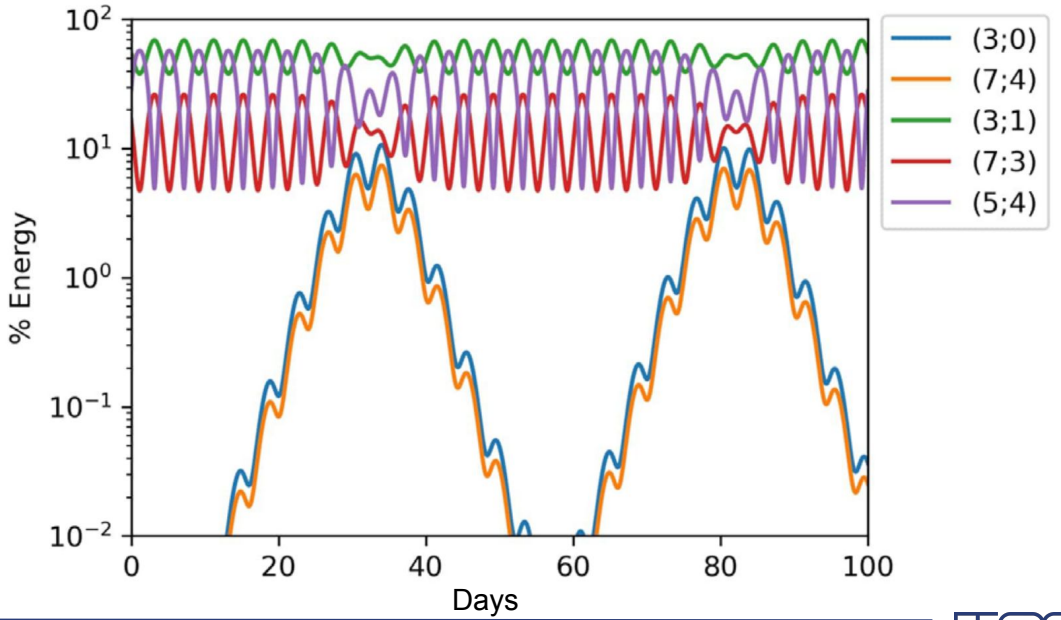




In this case, the inter triad precession energy transfer is small….

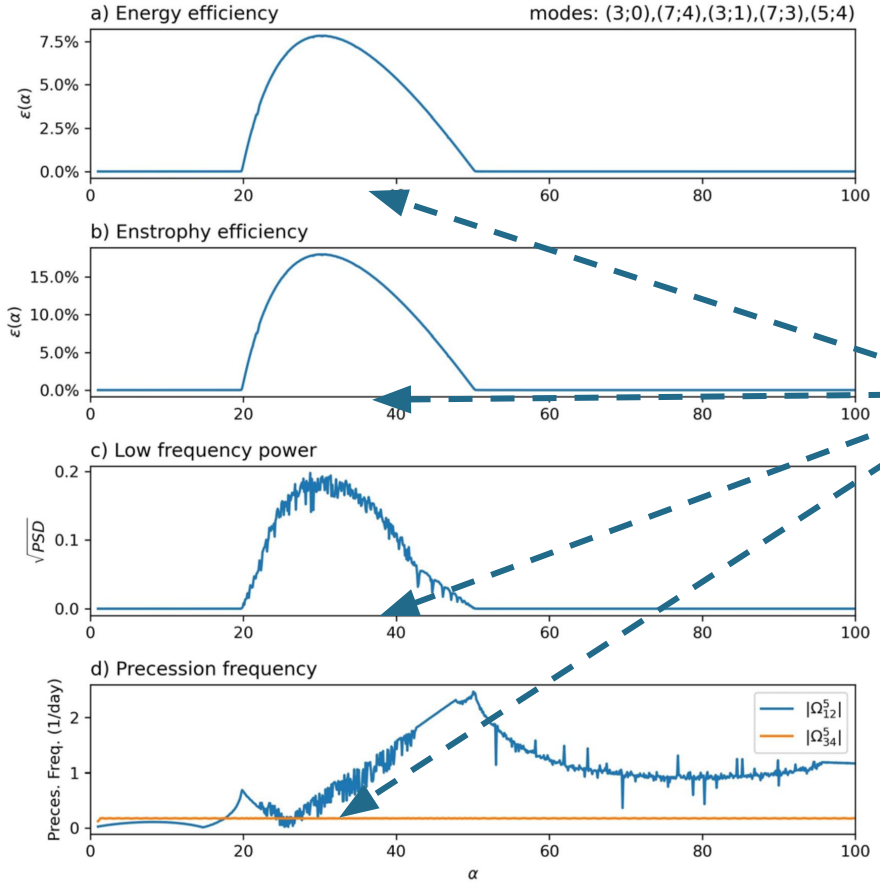
5 wave model (truncated ODE)





Efficient energy transfer between triads! Low frequency related!

5 wave model (truncated ODE)

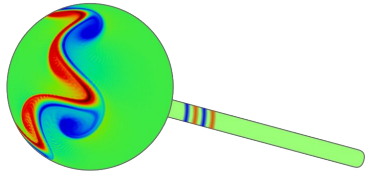


α Re-scales the initial conditions
(re-scaling the harmonic frequency)

Region of approximate **synchronization**
=> generates most **effective energy transfer** between triads!

Energy exchange of periods longer than 10 days!

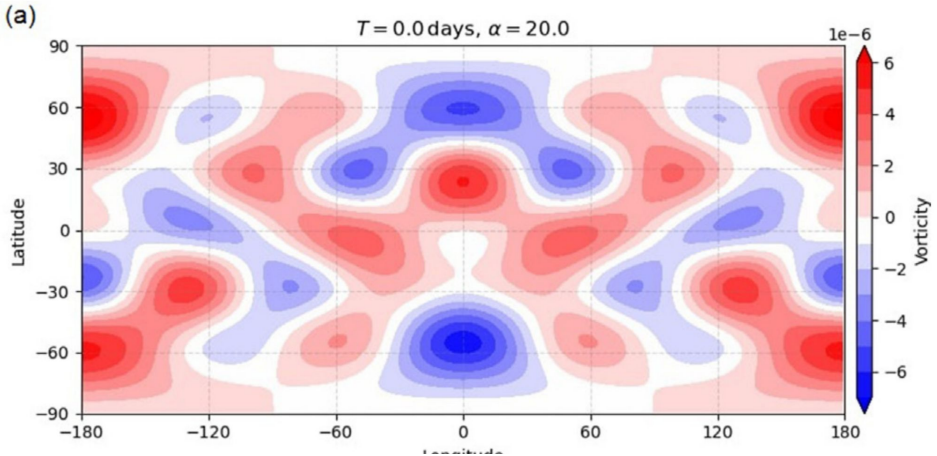
Analysis on the PDE model



- Barotropic Vorticity Equations on the Sphere
- SWEET software (Shallow Water Equation Environment for Tests (https://sweet.gitlabpages.inria.fr/sweet-www/)
- Spectral model (pseudo-spectral) with spherical harmonics

Initialize with energy on the main triad and observe energy transfers varying the intensity of the initial amplitudes (changing harmonic frequency and looking for precession resonance).

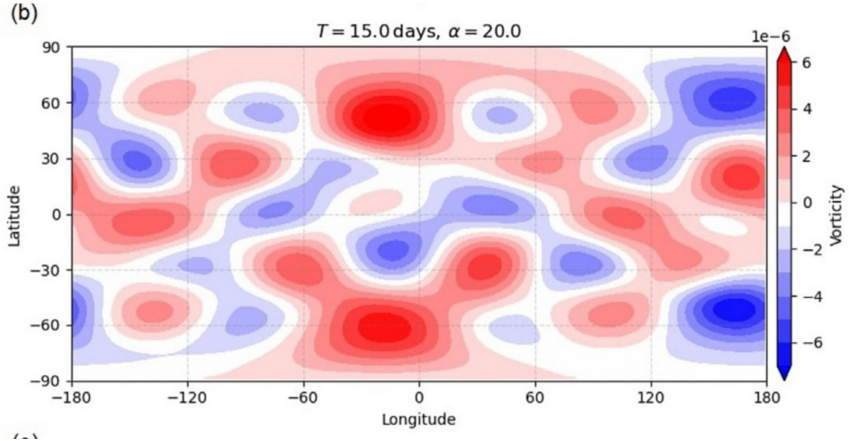
Example initializing in modes (5,4), (3,1), (7,3) :

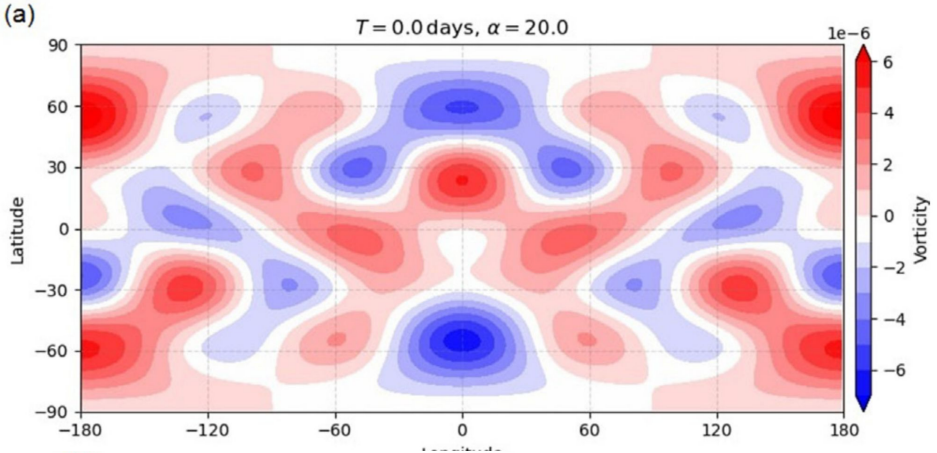


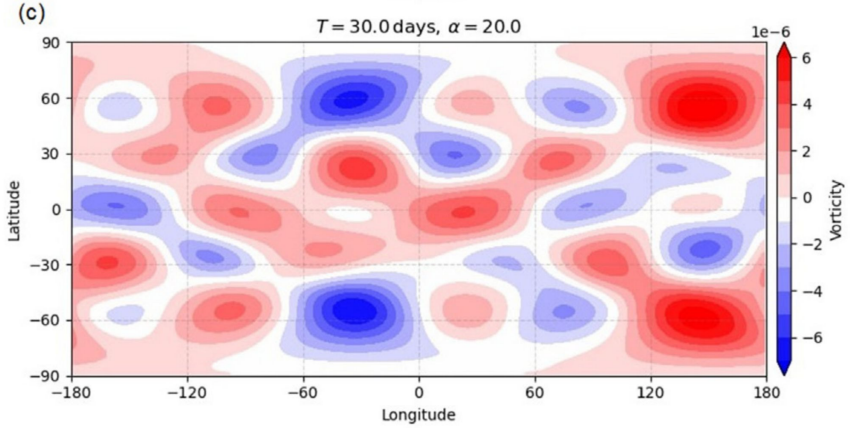
Analysis on the PDE model

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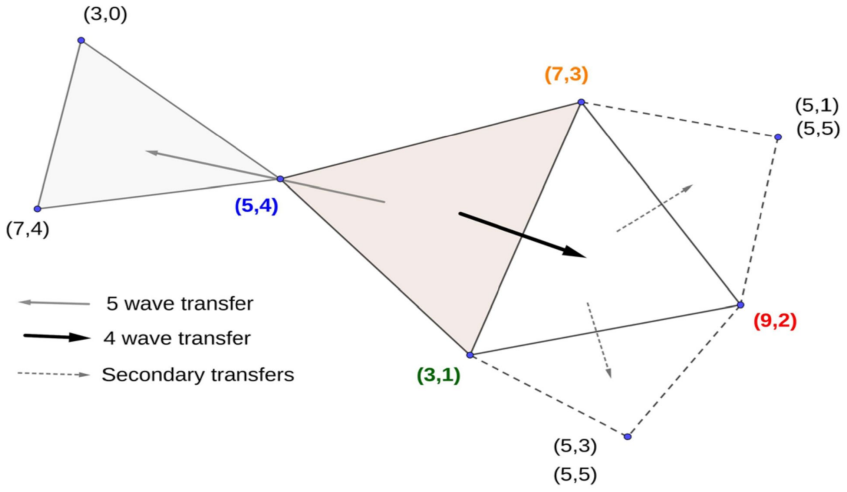


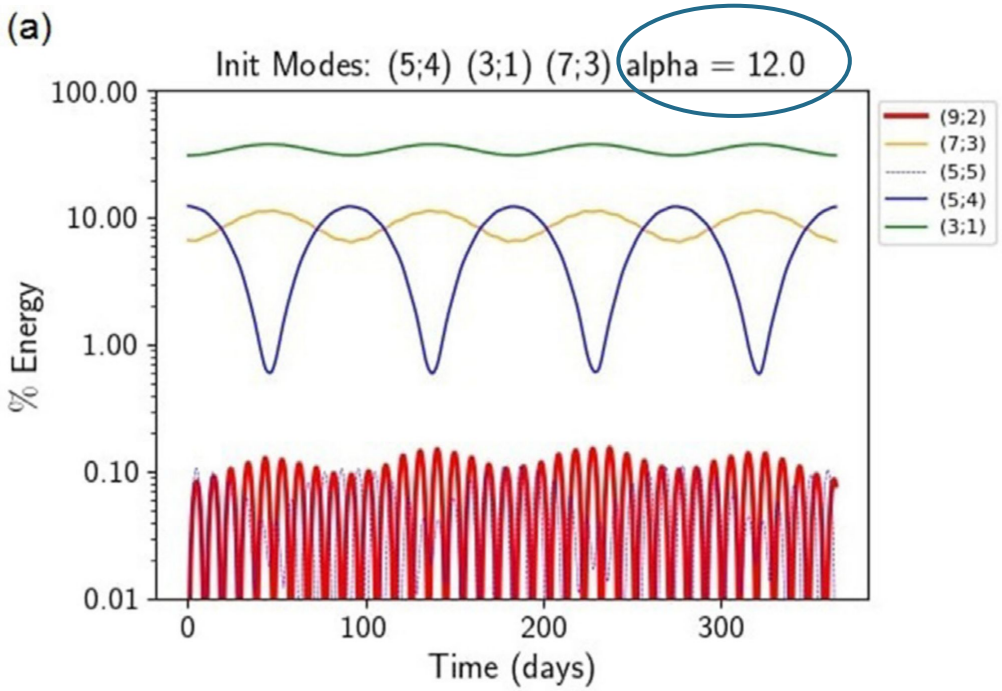


Analysis on the PDE model

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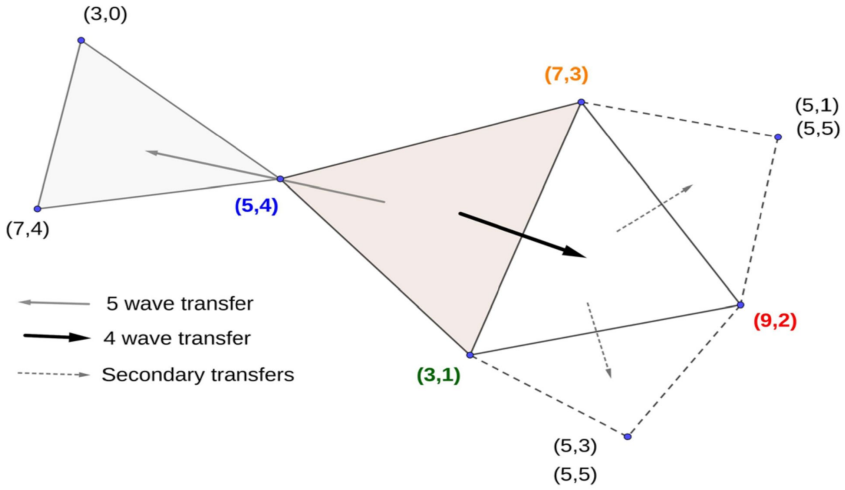


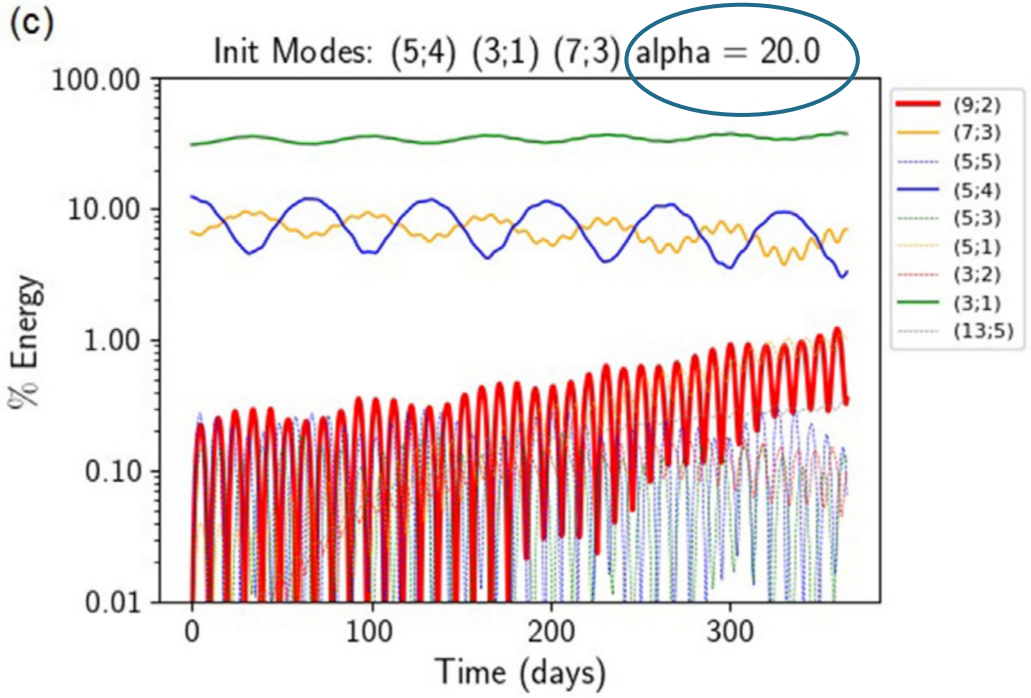
Systems essentially stays restricted to certain modes!

Analysis on the PDE model

Initialize with energy on the main triad and observe energy transfers varying the intensity of the initial amplitudes (changing harmonic frequency and looking for precession resonance).

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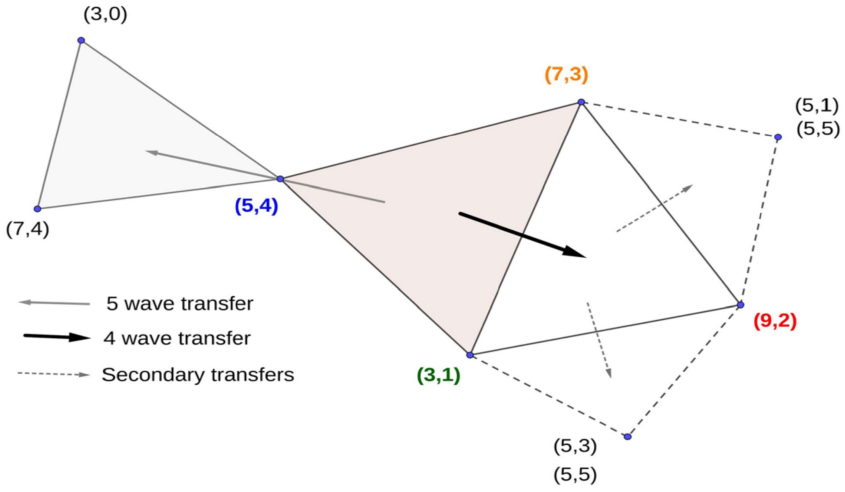


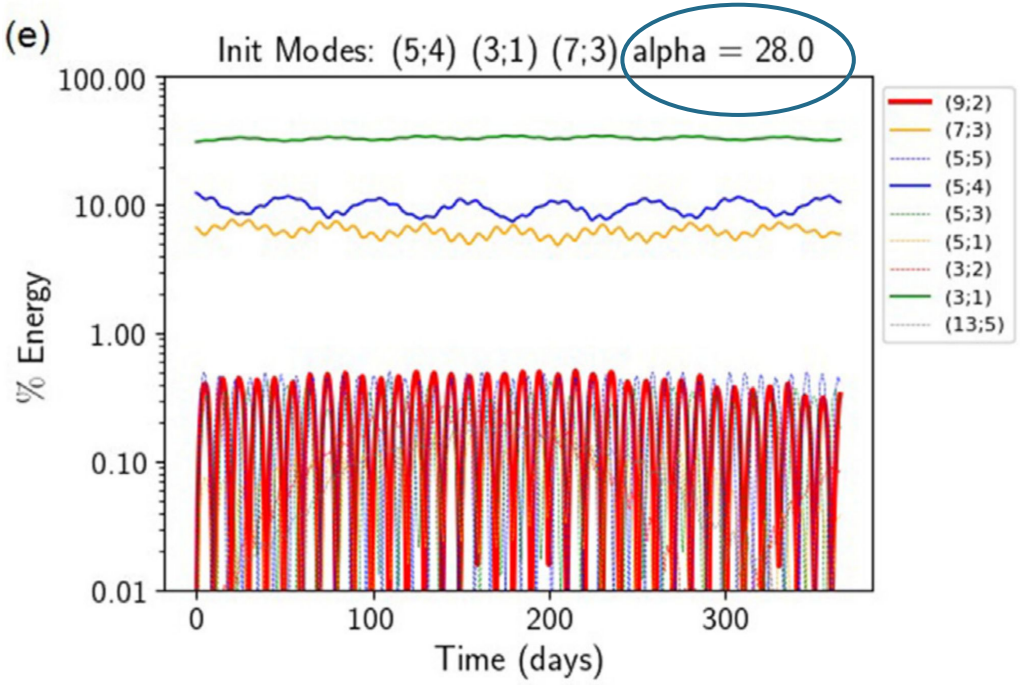
Efficient Inter triad energy exchanges!!

Analysis on the PDE model

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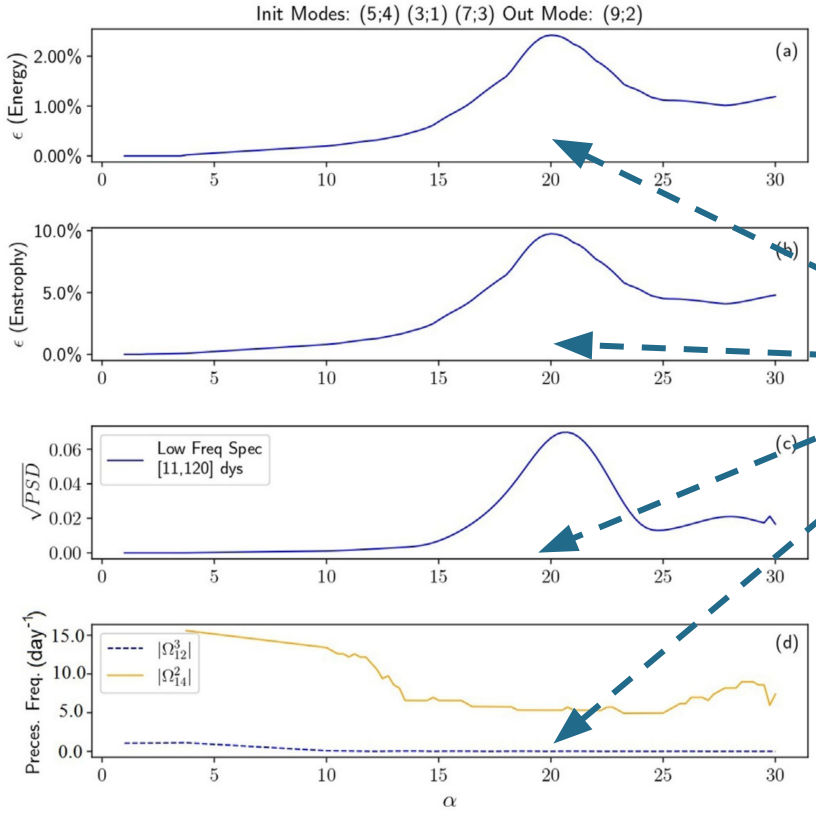
Example initializing in modes (5,4), (3,1), (7,3) :





Consistent energy exchanges between modes.

Analysis on the PDE model



α Re-scales the initial conditions
(re-scaling the harmonic frequency)

Region of approximate **synchronization**
=> generates most **effective energy transfer** between triads!

Energy exchange of periods longer than 10 days!

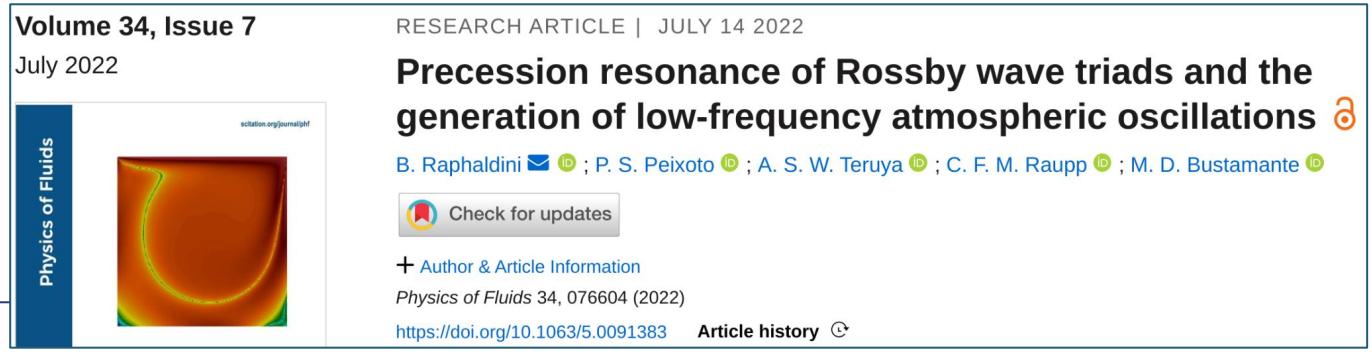
In summary

Atmospheric perspective:

- New perspective on **nonlinear inter triad interactions**, beyond simple triad resonances.
- Nonlinear triad interaction can explain **low frequency** (seasonal/sub-seasonal) variability in the atmosphere via inter triad energy transfer due to **phase alignment.** Quasi-Biennial Oscillation (QBO), Madden-Julian Oscilation (MJO) ?

Numerical modelling perspective:

- Complicated dynamics (phase alignments) that can be verified even in simple ODE models, which already capture well the dynamics observed in full PDE model!
- Eliminating certain high frequency waves can impact in important low frequency phenomena! Be careful with numerical methods…



Outlook

Atmospheric (ongoing):

- Extending inter triad precession theory to shallow water equations (include gravity waves in the interactions).
- Correlating with observational data - Quasi-Biennial Oscillation (QBO), Madden-Julian Oscilation (MJO) ?

Numerical modelling (ongoing):

- Coarse propagator in Parallel-in-time methods need to take into account more complex nonlinear interaction processes to capture low frequency oscillations and slow-fast interactions.

- Slow/fast splitting in numerical methods can kill certain relevant atmospherical phenomena. How to correctly split slow/fast? Is it possible?



“All models are wrong, but some are useful”

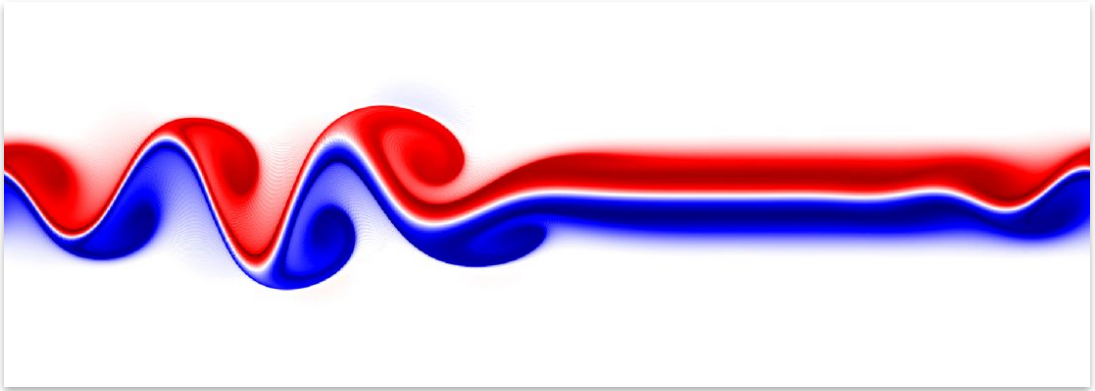
— George Box

Thanks!

More at: www.ime.usp.br/~pedrosp

Contact for collaboration/advisory: ppeixoto@usp.br

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- Many collaborators!
- FAPESP Jovem Pesquisador/BPE
- CNPq Universal/Produtividade
- CAPES Auxílios/Bolsas





PDEs

THE WORKSHOP ON PARTIAL DIFFERENTIAL EQUATIONS ON THE SPHERE



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INPE

MONAN

Model for Ocean-laNd-Atmosphere predictioN



Produce the best weather, climate, and environmental forecasting available worldwide for the South American region and adjacent oceans.

In the **MONAN work plan,** Brazil and Latin America adopt a unified and community Earth System model

All work on a single modeling system, a single computational code

1. **It involves the most relevant Earth System components:**
   a) Atmosphere, biosphere and continental soils, cryosphere, oceans, upper space.
   b) Anthropogenic disturbances relevant to the dynamics of the natural system (anthroposphere).
2. **Unified:**
   a) Suitable for spatial scales of atmospheric phenomena on the order of 10^2 m to 10^3 km.
   b) Suitable for temporal scales of now-casting, weather, sub-seasonal, seasonal and climate change.
3. **Accurate:**
   a) Reduced number of approximations and use of state-of-the-art numerical methods for solving differential equations in supercomputers.
   b) Anchored in a robust Data Assimilation System using remote sensing and local observations.
   c) Enhanced with Artificial Intelligence/Machine Learning techniques.
4. **Community model:**
   a) Open source with free and public access.
   b) Workshops and training for the community.
5. Multi-Institutional Governance:
   a) Scientific steering committee formed by representatives of operational and research centers, academy and stakeholders.

Local Organizers:

- Pedro da Silva Peixoto (Associate Professor at the Applied Maths Department - USP-Brazil) - www.ime.usp.br/~pedrosp
- Ariane Frassoni (Researcher at the Division for Earth System Numerical Modeling at the Brazilian National Institute for Space Research/Center for Weather Forecasting and Climate Studies - INPE/CPTEC - Brazil) https://www.researchgate.net/profile/Ariane-Frassoni
- Pedro Leite da Silva Dias (Full Professor - Meteorology - USP-Brazil) - https://scholar.google.com/citations?user=sJs905UAAAAJ&hl=en
- Saulo Ribeiro de Freitas (Head of the Earth System Modelling Division at the Brazilian National Institute for Space Research/Center for Weather Forecasting and Climate Studies - -INPE/CPTEC, Brazil) https://scholar.google.com/citations?hl=en&user=QOIDTywAAAAJ
- Haroldo Fraga de Campos Velho (Senior Researcher at INPE - National Institute for Space Research, Brazil) https://scholar.google.com.br/citations?user=OoYFpwkAAAAJ&hl=en

The local organization will also have the help of 3 secretariat personnel from the university, as well as several PhD and Postdoctoral researchers involved with MONAN and related projects.

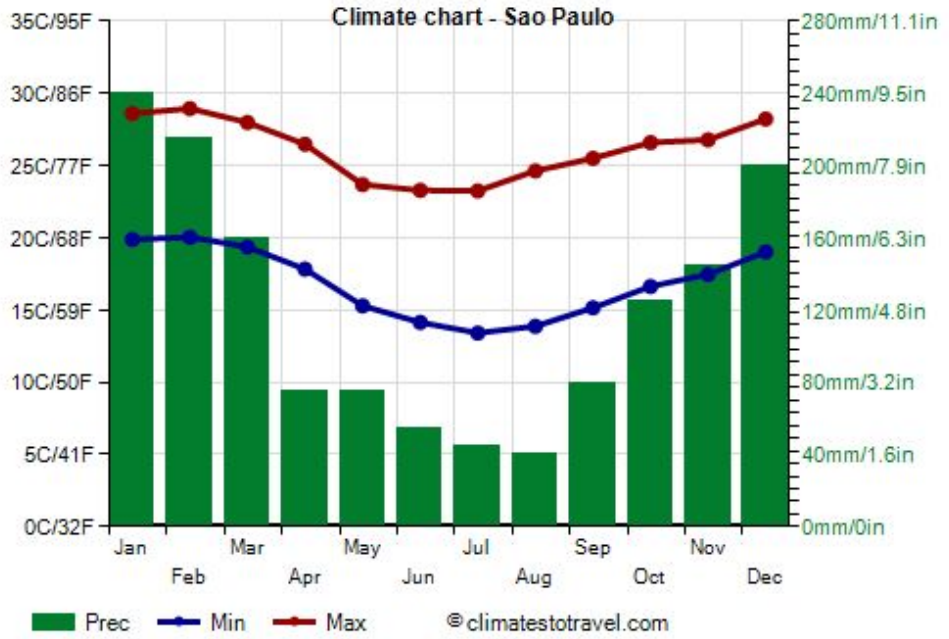
Funding:

- Possibility of applications for funding including air tickets + per diem (Brazilian agencies FAPESP, CNPq, CAPES, USP).

When? 2025?

Avoid:

- January/February (High season summer) - Crowded and rainy
- Carnaval (early March) - Fun but very very crowded
- August (vacations in Europe)



Suggestions?

- Late March 2025?
- September/October/November?

Where?

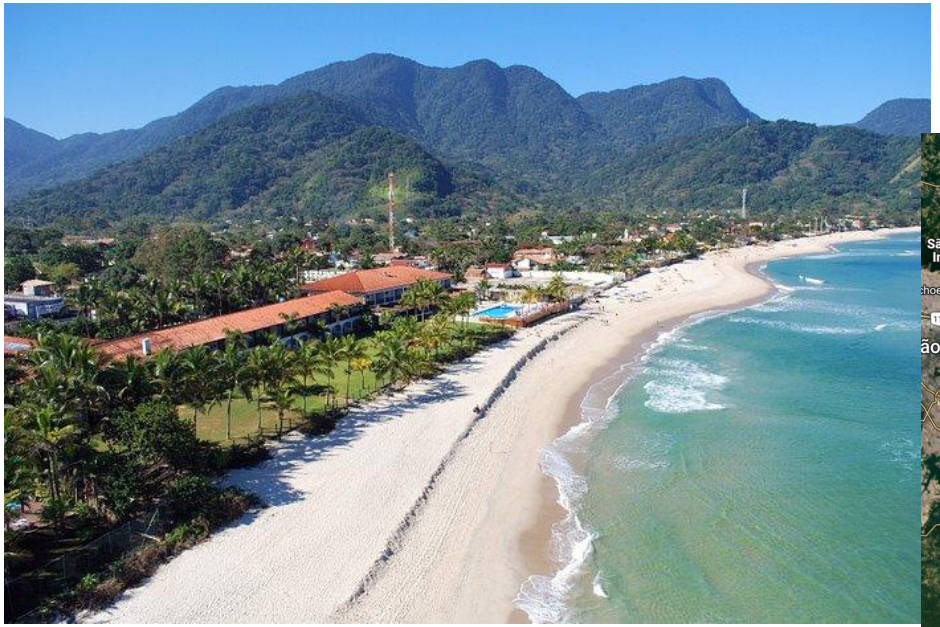
IME
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE DE SÃO PAULO



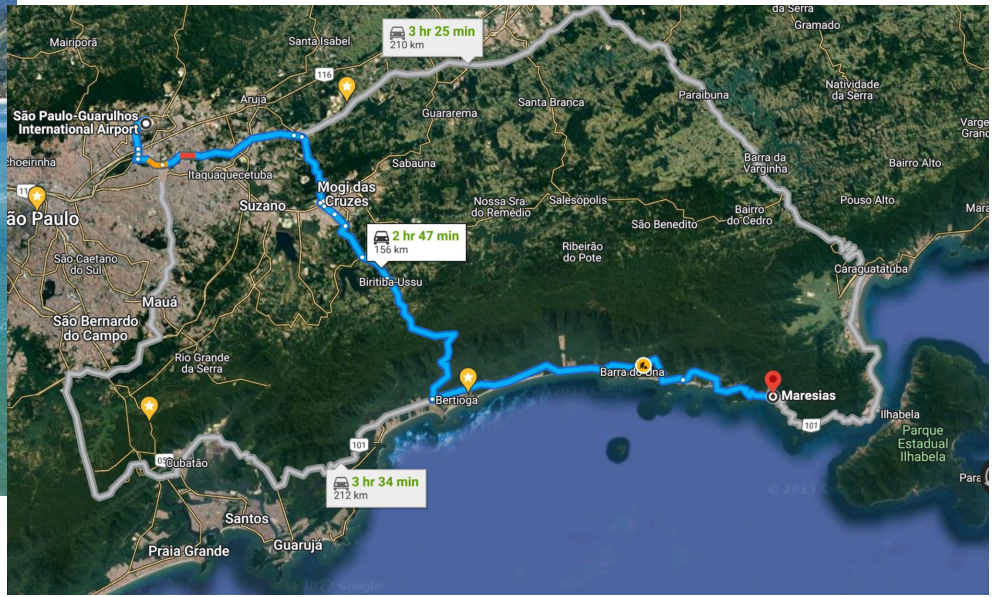








Beach resort?



- 4+ hour bus ride from the airport (3x a day)
- All inclusive (rooms, food, conference room, …)

Logistics

Direct flights



Carbon footprint….hybrid?

Obrigado!

Hope to see you all in Brazil!



PDEs

THE WORKSHOP ON PARTIAL DIFFERENTIAL EQUATIONS ON THE SPHERE

