# Stability of partially ordered sets and differentiation with respect to a suitable pair

Gustavo Peres Filbrich

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Supervisor: Prof. Dr. Kostiantyn Iusenko

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## Abstract

In the 1970s, Nazarova and Roiter developed a differentiation algorithm for partially ordered sets (posets) with respect to a maximal element, having Gabriel given it a categorical interpretation through his differentiation functor. Fonseca studied the behaviour of stable representations of posets, a concept coined by Iusenko and Futorny, under this maximal–element differentiation. She also related the Cartan matrices of a poset and of its differential. In a similar fashion, Zavadskij created a differentiation algorithm for posets with respect to a suitable pair of points, having given, himself, a categorical interpretation of it. In this work, we also study how Cartan matrices of a poset and of its differential are connected and how stability is affected, however through Zavadskij's suitable–pair differentiation procedure.



**Keywords:** posets, representations, differentiation, stability, Cartan matrix.



## Resumo

Nos anos 1970, Nazarova e Roiter desenvolveram um algoritmo de diferenciação para conjuntos parcialmente ordenados (posets) com respeito a um elemento maximal, tendo Gabriel dado a ele uma interpretação categórica através de seu funtor diferenciação. Fonseca estudou o comportamento de representações estáveis de posets, um conceito cunhado por Iusenko e Futorny, sob essa diferenciação maximal. Ela também relacionou as matrizes de Cartan de um poset e de seu diferencial. De maneira similar, Zavadskij criou um algoritmo de diferenciação de posets com respeito a um par apropriado de elementos, tendo dado, ele mesmo, uma interpretação categórica a ele. Neste trabalho, também estudamos como as matrizes de Cartan de um poset e de seu diferencial estão conectadas e como a estabilidade é afetada, no entanto através do procedimento com um par apropriado de Zavadskij.



**Palavras-chave:** posets, representações, diferenciação, estabilidade, matriz de Cartan.

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# Introduction

Finding relations between mathematical structures has been an incredibly powerful tool for solving problems. Not only are we able to approach problems through a better understood *language*, but also we may frequently simplify them. That's one of the main goals of *Representation Theory*.

In the context of resolving Brauer–Thrall's second conjecture, Nazarova and Roiter introduced *matrix representations* [NR75]. A *maximal–element* differentiation algorithm for partially ordered sets (posets) was also estabilished by them, allowing Kleiner to classify partially ordered sets of finite representation type [Kle75]. Categorical properties of the algorithm were then given by Gabriel, when *Subspace Representations of Posets* and a *Maximal–Element Differentiation Functor* were introduced by him [Gab73].

Aiming for applicability in a more general situation, Zavadskij defined a differentiation algorithm for posets with respect to a suitable pair of points [Zav77]. This latter procedure allowed Nazarova and him to classify posets of finite growth representation type [ZN82]. This current work is mostly devoted to analysing this algorithm and its associated differentiation map.

Classifying indecomposable modules of a given algebra may be a wild problem, in the sense of being essentially impossible. Using Mumford, Fogarty and Kirwan's geometrical approach [MFK94], King addressed the issue by considering spaces in which points naturally correspond to isomorphism classes of representations [Kin94]. It gave rise to the idea of stable representations.

Stability for arbitrary abelian categories was a concept coined by Rudakov [Rud97], when he described the conditions of stability for the representation theory of quivers. Futorny and Iusenko [FI19b] defined the notion of stability in the framework of representations of posets.

Fonseca's PhD thesis [Fon23] explores the connection between poset stability and the maximal–element differentiation algorithm, particularly envisioning that this differentiation procedure preserves stability. Not only has she presented a linear transformation that associates a subspace representation's dimension vector to its differential one, but also provided a method to calculate the way stability changes through the algorithm.

The goal of this work was to extend Fonseca's ideas to Zavadskij's differentiation algorithm. A similar approach was taken, having been developed a linear transformation that leads the dimension vector of a given representation to its corresponding one after a suitable–pair differentiation procedure.

Differently from the case of a maximal–element poset differentiation, it was not possible to algorithmize via linear transformations the manner a stability weight changes after suitable–pair differentiation. However, some lemmas help us build the new weight case–by–case.

Apendix A contains a free translation of Zavadskij's article defining his suitable–pair differentiation algorithm. The reason for including it in this work was the lack of an English translation from the original Russian version.

# Chapter 1

# Preliminaries

This starting chapter is devoted to establishing the ground upon which *Representation Theory of Posets* was built. A brief overview is intended, together with examples that led to the development of the apparatus we currently have. We begin by defining *Classification Problems*, as well as some important ideas strongly used by Nazarova, Roiter [NR75] and Gabriel [Gab73], such as *Matrix Problems* and *Partially Ordered Sets*. A glimpse at *Quivers* and *Path Algebras* is provided so to motivate the concept of *Stability*.

## 1.1. Classification Problems

Roughly speaking, classifying a mathematical object stands for its description up to some isomorphism. Introductory *Linear Algebra* courses already bring up such ideas when dealing with *similarity and equivalence of matrices*.

In order to better understand classification, some categorical terms will be required. We are following some of Simson's textbook remarks on the topic [Sim92, Chapter 1].

An additive category $\mathcal{B}$ is said to have a *finite unique decomposition property* if, for every object $X$ of $\mathcal{B}$, there exists a decomposition $X \simeq X_1 \oplus \cdots \oplus X_m$, where $X_1, \ldots, X_m$ are indecomposable. In addition, if one finds another decomposition $X \simeq X'_1 \oplus \cdots \oplus X'_r$, where $X'_1, \ldots, X'_r$ are equally indecomposable, then $r = m$ and there exists a permutation $\sigma : \{1, \ldots, m\} \longrightarrow \{1, \ldots, m\}$ such that $X_i \simeq X'_{\sigma(i)}$ for every $i = 1, \ldots, m$.

Given $R$ an associative ring with identity, it is quite usual for us to deal with additive subcategories of mod($R$), the category of finitely generated right–modules over $R$. In such cases, interesting queries naturally arise, such as

**i.** the explicit description of the indecomposable objects, or;

**ii.** given an object of the category, finding its decomposition into direct sum of indecomposable ones.

Problems admitting classification of its indecomposable objects are said to be of *tame* type, which may be of *finite* or *infinite* type. There are, however, problems which classification is virtually impossible, i.e. they contain the problem of classifying pairs of (noncommuting) matrices up to simultaneous similarity – these are said to be of *wild* type.

## 1.2. Matrix Problems

L. A. Nazarova and A. V. Roiter have successfully solved classification problems in some cases by reducing them to what was called *Matrix Problems*. A particularly important result they achieved using the method aforementioned was solving Brauer–Thrall's second conjecture [NR75].

Given a field $k$ we shall consider $\mathbb{M}(k)$ the set of finite matrices with entries in $k$. For matrices $A, B \in \mathbb{M}(k)$, their direct sum is defined as follows:

$$A \oplus B := \left[\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & B \end{array}\right],$$

where $\mathbf{0}$ is a suitable–sized matrix with zeroes in all of its entries. It may be effortlessly verified that $\mathbb{M}(k)$ is closed under this operation.

Now, we shall consider $\mathcal{G}$, a set of elementary transformations between rows and columns of matrices in $\mathbb{M}(k)$. We say that $A$ is $\mathcal{G}$–equivalent to $B$ (denoted $A \sim_{\mathcal{G}} B$) if $B$ may be obtained from $A$ through a series of transformations in $\mathcal{G} \cup \mathcal{G}^{-1}$.

A class $A/\sim_{\mathcal{G}}$, represented by the matrix $A \in \mathbb{M}(k)$, is said to be decomposable if there exist non–trivial matrices $B, C \in \mathbb{M}(k)$ such that $A \sim_{\mathcal{G}} B \oplus C$. Similarly, $A/\sim_{\mathcal{G}}$ is said to be indecomposable if $A \nsim_{\mathcal{G}} \mathbf{0}$ and $A$ isn't decomposable.

A *Matrix Problem* is, therefore, the problem of classifying indecomposables in the residue class $\mathbb{M}(k)/\sim_{\mathcal{G}}$. Likewise, it may be interpreted as the reduction of any matrix in $\mathbb{M}(k)$ to a canonical form through transformations in $\mathcal{G} \cup \mathcal{G}^{-1}$.

**Example 1.2.1 ([Sim92], Section 1.2).** *The* Classification Theorem of Finitely Generated Abelian Groups *may be understood as a Matrix Problem. Consider* $G \in mod(\mathbb{Z})$ *a finitely generated abelian group.*

*It means that there exist $g_1, \ldots, g_n \in G$ such that $G = \langle g_1, \ldots, g_n \rangle$. Now we take the following onto homomorphism into account:*

$$H : \mathbb{Z}^n \longrightarrow G$$
$$(z_1, \ldots, z_n) \longmapsto z_1 g_1 + \cdots + z_n g_n$$

*By the* First Isomorphism Theorem*, we get that $G \cong \mathbb{Z}^n / \ker(H)$. It is known that every subgroup of $\mathbb{Z}^n$ is finitely generated by at most $n$ generators, so let's consider $k_1, \ldots, k_m$ ($m \leq n$) generators of $\ker(H)$. Writing $k_i = (k_{i1}, \ldots, k_{in})$ for every $i = 1, \ldots, m$ leads us to the integral matrix $A \in \mathbb{M}_{m \times n}(\mathbb{Z})$:*

$$A = \begin{bmatrix} k_{11} & \ldots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{m1} & \ldots & k_{mn} \end{bmatrix}.$$

*Using elementary row and column transformations, the matrix $A$ can be reduced to its* Smith Normal Form*. In other words, there exist matrices $B \in SL_m(\mathbb{Z})$ and $C \in SL_n(\mathbb{Z})$ such that $D = C^{-1} A B$ is a diagonal matrix:*

$$D = \left[\begin{array}{cccc|ccc} d_1 & & & \mathbf{0} & 0 & \ldots & 0 \\ & \ddots & & & \vdots & \ddots & \vdots \\ & & d_s & & 0 & \ldots & 0 \\ & & & 0 & 0 & \ldots & 0 \\ & & & \ddots & \vdots & \ddots & \vdots \\ & \mathbf{0} & & & 0 & \ldots & 0 \end{array}\right],$$

*where $d_1, \ldots, d_n$ are non-zero positive integers. Finally, there exists a commutative diagram of abelian groups*

$$\begin{array}{ccccccc} \mathbb{Z}^m & \xrightarrow{A} & \mathbb{Z}^n & \longrightarrow & G & \longrightarrow & 0 \\ \cong\uparrow B & & \cong\uparrow C & & \cong\uparrow & & \\ \mathbb{Z}^m & \xrightarrow{D} & \mathbb{Z}^n & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

*and $G \cong N \cong \mathbb{Z}^{n-s} \oplus \mathbb{Z}/\langle d_1 \rangle \oplus \cdots \oplus \mathbb{Z}/\langle d_s \rangle$.*

## 1.3. Initial Terminology on Partially Ordered Sets

Interesting results have been achieved by Nazarova and Roiter when restricting the study of Matrix Problems to *Partially Ordered Sets* and their representations.

**Definition 1.3.1.** *A* partially ordered set*, abbreviated* poset*, consists of a set $\mathcal{S}$ together with a binary relation $\preceq$ satisfying the following axioms:*

***i.*** reflexivity: *$a \preceq a$ for every element $a \in \mathcal{S}$;*

***ii.*** antisymmetry: *if $a \preceq b$ and $b \preceq a$ for $a$ and $b$ elements of $\mathcal{S}$, then $a = b$;*

***iii.*** transitivity: *if $a \preceq b$ and $b \preceq c$ for $a, b$ and $c$ elements of $\mathcal{S}$, then $a \preceq c$.*

It follows from the definition that not every two elements of a poset $(\mathcal{S}, \preceq)$ are necessarily comparable. For the sake of simplicity, we write $\mathcal{S}$ for the poset $(\mathcal{S}, \preceq)$ whenever the partial order relation $\preceq$ is implicitly understood. Moreover, throughout this text, we shall assume that $\mathcal{S}$ denotes a finite poset and that $k$ is a fixed field.

**Definition 1.3.2.** *A poset $\mathcal{S}$ may be equivalently depicted in other ways rather than the explicit description of its order relations. Namely, they consist of*

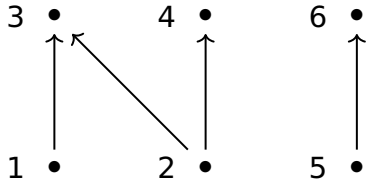
***i.*** *its* Hasse Diagram*, the diagram whose vertices are the elements of $\mathcal{S}$ and, given $a, c \in \mathcal{S}$, there is an arrow $a \bullet \longrightarrow \bullet\ c$ if $a \prec c$ and there is no other element $b \in S$ such that $a \prec b \prec c$;*

***ii.*** *its* Cartan Matrix*, the square integral matrix $C_{\mathcal{S}} \in \mathbb{M}_{|\mathcal{S}|\times|\mathcal{S}|}(\mathbb{Z})$ given by*

$$(C_{\mathcal{S}})_{ij} = \begin{cases} 1, \text{ if } i \preceq j; \\ 0, \text{ otherwise.} \end{cases}$$

**Example 1.3.3.** *The poset $\mathcal{S} = \{1, 2, 3, 4, 5, 6 \mid 1 \prec 3,\ 2 \prec 3,\ 2 \prec 4,\ 5 \prec 6\}$ may be portrayed as the following Hasse Diagram and Cartan Matrix:*

$\mathcal{S}$ :




$$C_{\mathcal{S}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

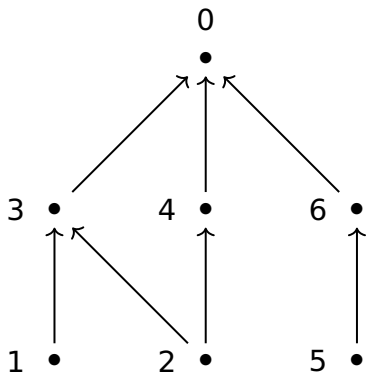
*Due to the shape of its Hasse diagram, this poset $\mathcal{S}$ is also known as* $(N, 2)$.

Given the *Cartan Matrix* $C_{\mathcal{S}}$ of a finite poset $\mathcal{S}$, there will always exist its inverse matrix $C_{\mathcal{S}}^{-1}$. This fact can be easily checked by observing that the elements of $\mathcal{S}$ may be labelled in a way that $i \leq_{\mathbb{Z}} j$, in the natural ordering of integers, when $i \preceq j$. Through this convenient labelling, the *Cartan Matrix* of $\mathcal{S}$ turns out to be an upper–triangular matrix whose elements of the main diagonal are all ones and $\det C_{\mathcal{S}} = 1 \neq 0$.

Throughout this work we shall often need the enlargement of a poset $\mathcal{S}$ by a unique maximal element 0, being denoted $\mathcal{S}* = \mathcal{S} \cup \{0\}$.

**Example 1.3.4.** *If* $\mathcal{S} = (N, 2)$ *from* Example 1.3.3*, its enlargement* $\mathcal{S}*$ *by a maximal element* 0 *is*

$\mathcal{S}*$ :




It is common to omit the arrows pointing at $\{0\}$ in order to avoid over–complicating the diagram.

## 1.4. Matrix Representations of Posets

Following a somehow chronological order of events, we now establish an important connection between *Matrix Problems* and *Partially Ordered Sets*, a link set up by Nazarova and Roiter [NR75].

**Definition 1.4.1.** *Let $\mathcal{S} = \{1, \ldots, n\}$ be a finite poset and $s_0, s_1, \ldots, s_n \in \mathbb{N}$. Given matrices $A_1 \in \mathbb{M}_{s_0 \times s_1}(k), \ldots, A_n \in \mathbb{M}_{s_0 \times s_n}(k)$, a* Matrix Representation *of $\mathcal{S}$ is a block matrix $A$ of the form*

$$A = \big[ \underbrace{A_1}_{s_1} \mid \underbrace{A_2}_{s_2} \mid \ldots \mid \underbrace{A_n}_{s_n} \big] \big\}_{s_0} .$$

Given matrix representations $A_1$ and $A_1'$ of $\mathcal{S}$, we may define the direct sum between them:

$$A \oplus A' = \left[\begin{array}{c|c|c|c|c} A_1 & \mathbf{0} & & A_n & \mathbf{0} \\ \hline \mathbf{0} & A_1' & \ldots & \mathbf{0} & A_n' \end{array}\right] .$$

**Remark 1.4.2.** *The set $\mathbb{M}_{\mathcal{S}}$ of all matrix representations of $\mathcal{S}$ is closed under the direct sum mentioned above.*

The set of *compositions of elementary transformations* is denoted by $\mathcal{G}_{\mathcal{S}}$ and it consists of the following:

**i.** all simultaneous elementary transformations between rows;

**ii.** all elementary transformations between columns from a same block;

**iii.** all elementary transformations adding a column of the $i^{th}$ block to the $j^{th}$ block, provided that $i \preceq j$.

With the notation aforementioned, a poset $\mathcal{S}$ is said to be of *finite representation type* if the associated matrix problem $(\mathbb{M}_{\mathcal{S}}, \mathcal{G}_{\mathcal{S}})$ is so.

**Example 1.4.3.** *If $\mathcal{S}$ is the poset $(N, 2)$ from* Example 1.3.3, *the following block matrix $A \in \mathbb{M}_{3\times 6}(\mathbb{Z})$ is an example of a matrix representation of $\mathcal{S}$.*

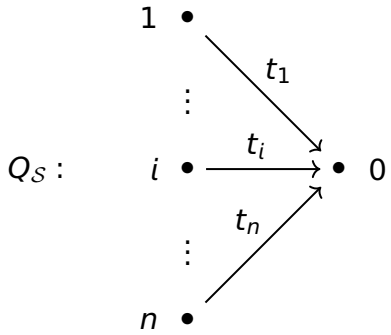
$$A = \left[\begin{array}{c|c|c|c|c|c} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

It turned out to be advantageous using the *additive enlargement* of $\mathbb{M}_{\mathcal{S}}$, known as the *Additivisation* of $\mathbb{M}_{\mathcal{S}}$ and denoted $\mathbb{M}_{\mathcal{S}}^{ad}$. Given a finite poset $\mathcal{S}$, where $|\mathcal{S}| = n \in \mathbb{N}$, it consists of the category in which

**i.** the objects are defined as systems

$$\boldsymbol{V} = (V_0, V_1, \ldots, V_n ; t_1, \ldots, t_n) \, ,$$

where $V_0, V_1, \ldots, V_n$ are finite–dimensional $k$–vector spaces and $t_i : V_i \longrightarrow V_0$ are $k$–linear transformations for all $i \in \mathcal{S}$. $\boldsymbol{V}$ may also be interpreted as a representation of the following quiver:



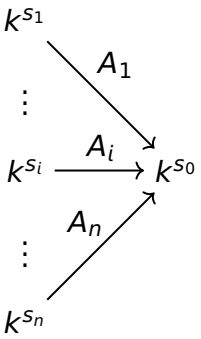

**ii.** given objects $\boldsymbol{V}$ and $\boldsymbol{V'}$ in $\mathbb{M}_{\mathcal{S}}^{ad}$, a morphism between them is a pair $(f, f_0)$ of $k$–linear maps which make the following diagram commute:

$$\begin{array}{ccc} V_1 \oplus \cdots \oplus V_n & \xrightarrow{(t_i)} & V_0 \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f_0} \\ V'_1 \oplus \cdots \oplus V'_n & \xrightarrow[(t'_i)]{} & V'_0 \end{array}$$

From now on, we begin to estabilish important connections that will later lead us to the results obtained. We aren't delving much into the topic as the approach intended in this particular section is merely descriptive. Further reading may be encountered at Simson's textbook [Sim92, Chapter 2.2].

A quite useful embedding of categories $\boldsymbol{i} : \mathbb{M}_{\mathcal{S}} \longrightarrow \mathbb{M}_{\mathcal{S}}^{ad}$ is hereby defined. Given a block matrix $A = [A_1|\dots|A_n] \in \mathbb{M}_{\mathcal{S}}(k)$, where $A_i \in \mathbb{M}_{s_i \times s_0}(k)$ for all $i \in \mathcal{S}$, it is defined

$\boldsymbol{i}(A)$ :




being $A_i : k^{s_i} \longrightarrow k^{s_0}$ the $k$–linear map induced by the matrix $A_i$ in the canonical bases .

Not only can we check that $\mathbb{M}_{\mathcal{S}}^{ad}$ is additive and has the finite unique decomposition property, but also we may obtain the following result.

**Lemma 1.4.4 ([Sim92], Lemma 2.2).** *Let $A, B \in \mathbb{M}_{\mathcal{S}}$ be block matrices and $\mathcal{G}_{\mathcal{S}}$ be a set of compositions of elementary transformations.*

***i.*** *$A \sim_{\mathcal{G}} B \iff \boldsymbol{i}(A) \cong \boldsymbol{i}(B)$.*

***ii.*** *$A$ is indecomposable in $(\mathbb{M}_{\mathcal{S}}, \mathcal{G}_{\mathcal{S}}) \iff \boldsymbol{i}(A)$ is indecomposable in $\mathbb{M}_{\mathcal{S}}^{ad}$.*

*Sketch of Proof.*

**i.** The block matrix $A$ may be understood as a matrix with $s_1 + \cdots + s_n$ rows and $s_0$ columns. From the definition of $\mathcal{G}_{\mathcal{S}}$, we can gather that elementary transformations between rows of $A$ correspond to left multiplication of $A$ by invertible matrices in $\mathrm{GL}_{s_0}(k)$.

We also see that elementary transformations between columns within a same block or adding a column from the $i^{th}$ block to the $j^{th}$ block, as long as $i \preceq j$, may be encoded as right–multiplication of $A$ by matrices $B \in \mathrm{GL}_{s_1+\cdots+s_n}(k)$ of the form

$$B = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ & B_{22} & \dots & B_{2n} \\ & & \ddots & \vdots \\ \mathbf{0} & & & B_{nn} \end{bmatrix},$$

where $B_{ii} \in \mathrm{GL}_{s_i}(k)$ and $B_{ij} = 0$ if $i \not\preceq j$ for all $i, j \in \mathcal{S}$.

**ii.** The second statement can be proven by contraposition, assuming that either $A$ or $\boldsymbol{i}(A)$ are decomposable and applying the result from item **i.**

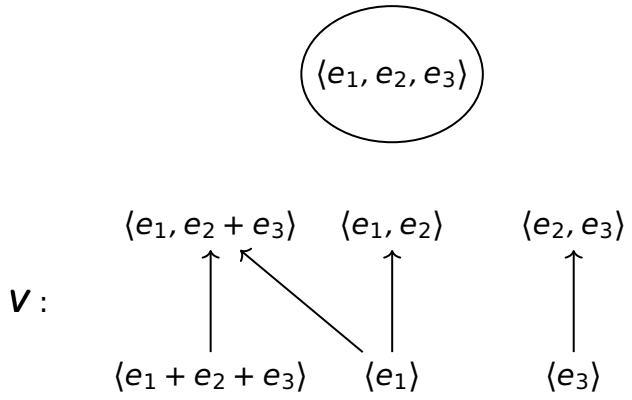
□

## 1.5. Subspace Representations of Posets

Most results presented and obtained in this work derive from *subspace representations of posets*, an idea coined by Gabriel [Gab73]. A functor plays a crucial role in relating them to *matrix representations of posets*.

**Definition 1.5.1.** *A* subspace representation *of $\mathcal{S}$ is a system $\boldsymbol{V} = (V_0 ; V_i)_{i \in \mathcal{S}}$ consisting of a finite dimensional $k$-vector space $V_0$, called the* ambient space *of $\boldsymbol{V}$, and subspaces $V_i$ of $V_0$ such that $V_i \subseteq V_j$ if $i \preceq j$.*

**Example 1.5.2.** *Let $\mathcal{S} = (N, 2)$ from* Example 1.3.3*. Fixing a basis $\{e_1, e_2, e_3\}$ for the vector space $k^3$, here follows an illustration for a representation $\boldsymbol{V}$ of $\mathcal{S}$:*




Following Arnold's textbook [Arn00, Section 1.2], we define the category of all subspace representations of a given poset $\mathcal{S}$, denoted $\mathfrak{sp}_{\mathcal{S}}$. Its objects consist of the previously seen systems of the form $\boldsymbol{V} = (V_0; V_i)_{i \in \mathcal{S}}$. Given objects $\boldsymbol{U} = (U_0; U_i)_{i \in \mathcal{S}}$ and $\boldsymbol{W} = (W_0; W_i)_{i \in \mathcal{S}}$, a morphism (also called a *representation*

*morphism*) is a $k$–linear transformation $f : U_0 \longrightarrow W_0$ such that $f(U_i) \subseteq W_i$ for all $i \in \mathcal{S}$. One observes that $\mathfrak{sp}_{\mathcal{S}}$ is an additive category [Arn00, Lemma 1.2.1] with a direct sum given by $\boldsymbol{U} \oplus \boldsymbol{W} = (U_0 \oplus W_0; U_i \oplus W_i)_{i \in \mathcal{S}}$.

We denote by $\mathrm{Ind}(\mathfrak{sp}_{\mathcal{S}})$ a set of representatives of the isomorphism classes of indecomposable objects of $\mathfrak{sp}_{\mathcal{S}}$. $\mathcal{S}$ is said to have *finite representation type* if $\mathrm{Ind}(\mathfrak{sp}_{\mathcal{S}})$ is finite and *infinite representation type* if $\mathrm{Ind}(\mathfrak{sp}_{\mathcal{S}})$ is infinite.

In the mid–1970s, Kleiner successfully classified all posets of finite representation type. His striking result had been originally proven by means os Nazarova and Roiter's concept of poset differentiation – an idea presented in the upcoming chapter. An alternative proof using Zavadskij's analogous construction is given in Chapter 3.

**Theorem 1.5.3 ([Kle75]).** *Let $\mathcal{S}$ be a finite poset. Then $\mathfrak{sp}_{\mathcal{S}}$ is of finite representation type if and only if $\mathcal{S}$ contains none of the following critical subposets:*

$\mathcal{K}_1 = (1,1,1,1)$ : $\mathcal{K}_2 = (2,2,2)$ :

$\mathcal{K}_3 = (1,3,3)$ :

$\mathcal{K}_4 = (N,4)$ : $\mathcal{K}_5 = (1,2,5)$ :

**Example 1.5.4.** *Although a proof of Theorem 1.5.3 is outlined in Section 3.3, we may effortlessly check that the poset $\mathcal{K}_1$ is of infinite representation type if we assume that the field $k$ is infinite. In order to do so, we build an indecomposable matrix representation $A^\lambda \in \mathbb{M}_{\mathcal{K}_1}$ for $\lambda \in k$:*

$$A^\lambda = \left[\begin{array}{c|c|c|c} \lambda & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right].$$

*Being $k$ infinite, the set $\{A^\lambda\}_{\lambda\in k}$ is also infinite.*

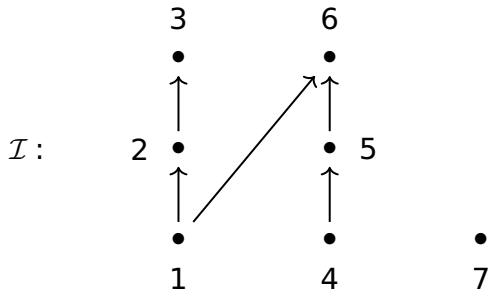
We may now define the reduction functor $H : \mathbb{M}_{\mathcal{S}}^{ad} \longrightarrow \mathfrak{sp}_{\mathcal{S}}$, which successfully links matrix representations to subspace representations of posets. It is built as follows:

**i.** given an object $\boldsymbol{V} = (V_0, V_i; t_i)_{i\in\mathcal{S}}$ in $\mathbb{M}_{\mathcal{S}}^{ad}$, we set $H(\boldsymbol{V}) = (M_0; M_i)_{i\in\mathcal{S}}$, where

$$M_0 = V_0,$$
$$M_j = \mathrm{Im}\left(\bigoplus_{i\preceq j} V_i \xrightarrow{(t_i)} V_0 = M_0\right) \text{ and;}$$

**ii.** given a morphism $(g; g_0) : \boldsymbol{V'} \longrightarrow \boldsymbol{V}$, then $g_0(M'_j) \subseteq M_j$ for all $j \in \mathcal{S}$. Therefore, we may set $H(g, g_0) = g_0$.

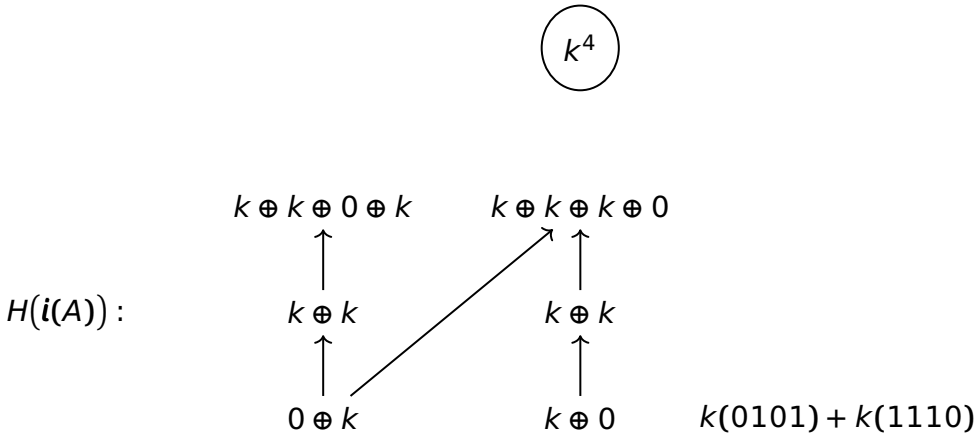
**Example 1.5.5 ([Sim92], Example 3.11).** *Let's consider the poset*




*and the matrix representation*

$$A = \left[\begin{array}{c|c|c|c|c|c|c|c} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \in \mathbb{M}_{\mathcal{I}}.$$
$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$$

*The functor H yields*




Before moving on, some categorical definitions are introduced.

**Definition 1.5.6.** *A covariant functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ yields an equivalence of categories if it is, simultaneously:*

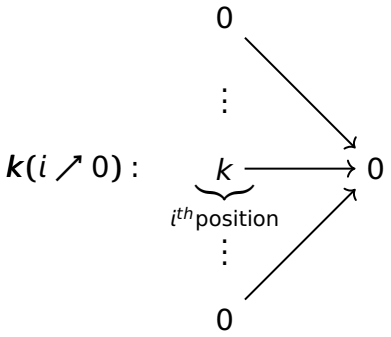
- ***i.*** *full, i.e. for every two objects $A_1, A_2$ of $\mathcal{A}$, the map $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(F(A_1), F(A_2))$ induced by F is surjective;*

- ***ii.*** *faithful, i.e. for every two objects $A_1, A_2$ of $\mathcal{A}$, the map $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(F(A_1), F(A_2))$ induced by F is injective;*

- ***iii.*** *essentially surjective (or dense), i.e. for each object B in $\mathcal{B}$ there exists an object A in $\mathcal{A}$ such that B is isomorphic to $F(A)$.*

**Definition 1.5.7.** *An additive functor F between additive categories is said to be a* representation equivalence *if it's full, dense and reflects isomorphisms, i.e. $F(f)$ is an isomorphism $\iff$ $f$ is an isomorphism.*

For the sake of simplicity, we'll refer to $\boldsymbol{k}(i \nearrow 0)$ as the indecomposable object in $\mathbb{M}_{\mathcal{S}}^{ad}$ given by




We now head into this section's main theorem, the ultimate connection between studying representations through matrices and subspaces.

**Theorem 1.5.8 ([Sim92], Theorem 3.1).** *Let $\mathcal{S} = \{1, \ldots, n\}$ be a poset and let $H$ be the reduction functor. Then,*

- ***i.*** *$H$ is full and dense;*
- ***ii.*** *given $\boldsymbol{V} \in \mathbb{M}_{\mathcal{S}}^{ad}$ indecomposable, then $H(\boldsymbol{V}) = 0 \iff \boldsymbol{V} \cong \boldsymbol{k}(i \nearrow 0)$ for some $i \in \mathcal{S}$;*
- ***iii.*** *consider $(\mathbb{M}_{\mathcal{S}}^{ad})_0$ the full subcategory of $\mathbb{M}_{\mathcal{S}}^{ad}$ consisting of objects having no summands of the form $\boldsymbol{k}(1 \nearrow 0), \ldots, \boldsymbol{k}(n \nearrow 0)$. Then $H$ induces a representation equivalence*

$$H_0 : (\mathbb{M}_{\mathcal{S}}^{ad})_0 \longrightarrow \mathfrak{sp}_{\mathcal{S}}$$

*and an equivalence of categories*

$$\mathbb{M}_{\mathcal{S}}^{ad} / [\boldsymbol{k}(1 \nearrow 0), \ldots, \boldsymbol{k}(n \nearrow 0)] \cong \mathfrak{sp}_{\mathcal{S}} \, ,$$

*where $[\boldsymbol{k}(1 \nearrow 0), \ldots, \boldsymbol{k}(n \nearrow 0)]$ is the two–sided ideal made of all maps that factorise through a direct sum of copies of $\boldsymbol{k}(1 \nearrow 0), \ldots, \boldsymbol{k}(n \nearrow 0)$.*

As a straightforward consequence of the previous theorem, we obtain the following result.

**Corollary 1.5.9.** *$\mathbb{M}_{\mathcal{S}}^{ad}$ is of finite representation type if and only if $\mathcal{S}$ contains none of Kleiner's critical posets $\mathcal{K}_1, \ldots, \mathcal{K}_5$ as a subposet.*

## 1.6. Quivers and path algebras

So far, we've dealt with representation theory of algebras through combinatorial approaches led by Nazarova, Roiter ([NR75]) and Gabriel ([Gab73]). However, the subject has also been addressed geometrically as we shall see in the following two sections. An *en passant* view is intended so to clear up the roots of this latter technique.

**Definition 1.6.1.** *A* quiver *is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of*

***i.*** *a set $Q_0$ of* vertices*;*

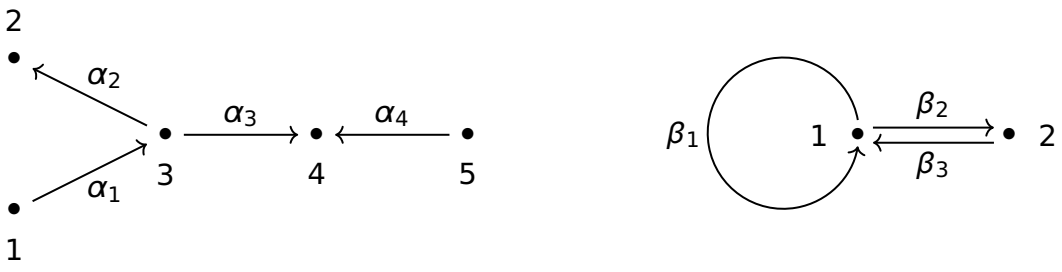
***ii.*** *a set $Q_1$ of* arrows*;*

***iii.*** *a map $s : Q_1 \longrightarrow Q_0$ that associates the* source *$s(\alpha) \in Q_0$ of every arrow $\alpha \in Q_1$ and;*

***iv.*** *a map $t : Q_1 \longrightarrow Q_0$ that associates the* target *$t(\alpha) \in Q_0$ of every arrow $\alpha \in Q_1$.*

We may refer to $Q = (Q_0, Q_1, s, t)$ simply as $Q$ whenever the sets and maps are implicitly understood. We may also write $a \xrightarrow{\alpha} b$ for an arrow $\alpha \in Q_1$ such that $s(\alpha) = a$, $t(\alpha) = b \in Q_0$.

**Example 1.6.2.** *Here are some examples of quivers. Note that, differently from posets, there may be cycles or multiple arrows with same sources and targets.*



Given two vertices $a, b \in Q_0$, a *path* from $a$ to $b$ of length $m \geq 1$ is a sequence $(a \mid \alpha_1, \alpha_2 \ldots, \alpha_m \mid b)$ where $\alpha_i \in Q_1$ for all $i = 1, \ldots, m$, $s(\alpha_1) = a$, $t(\alpha_m) = b$ and $t(\alpha_j) = s(\alpha_{j+1})$. It may be also portrayed as $\alpha_1\alpha_2 \ldots \alpha_m$ or as

$$a = a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_m} a_m = b$$

The set of all paths with length $l$ is denoted by $Q_l$. With this definition, we may associate a path of length $l = 0$ to each point $a \in Q_0$, the *trivial path* at $a$ and denoted by $\epsilon_a = (a \mid\mid a)$.

**Definition 1.6.3.** *Given a quiver $Q$, the* path algebra *of $Q$ is the $k$–algebra $kQ$ having the set of all paths $(a \mid \alpha_1, \ldots, \alpha_l \mid b)$ of length $l \geq 0$ in $Q$ as its basis. The product between two basis vectors $(a \mid \alpha_1, \ldots, \alpha_l \mid b)$ and $(c \mid \beta_1, \ldots, \beta_m \mid d)$ is given by*

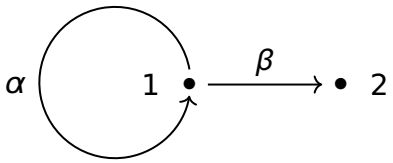
$$(a \mid \alpha_1, \ldots, \alpha_l \mid b)(c \mid \beta_1, \ldots, \beta_m \mid d) = \begin{cases} (a \mid \alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_m \mid d), \text{ if } t(\alpha_l) = s(b_1) \\ 0, \text{ otherwise.} \end{cases}$$

A quiver $Q$ is said to be *connected* if for every decomposition of the set of its vertices $Q_0 = Q_0' \cup Q_0''$, with both $Q_0'$ and $Q_0''$ non-empty, there exists an arrow $\alpha$ such that either $s(\alpha) \in Q_0'$ and $t(\alpha) \in Q_0''$ or $s(\alpha) \in Q_0''$ and $t(\alpha) \in Q_0'$.

If $Q$ is a finite and connected quiver, we may define the *arrow ideal of $Q$*, the two–sided ideal of the path algebra $kQ$ generated by the arrows of $Q$ and denoted by $R_Q$. Similarly, we may define $R_Q^m$ as the ideal of $kQ$ generated by the set of all paths of length at $m \geq l$.

A two–sided ideal $\mathcal{I}$ of $kQ$ is called *admissible* if there's $m \geq 2$ satisfying $R_Q^m \subseteq \mathcal{I} \subseteq R_Q^2$. It is, however, more usual to define admissible ideals in terms of *relations*, i.e. $k$–linear combinations of paths of length at least two having same source and target.

**Example 1.6.4.** *Consider the following quiver Q.*

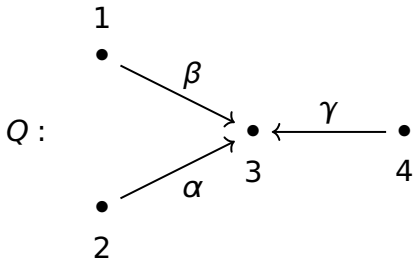



*The ideal* $\mathcal{I} = \langle \beta^3, \alpha\beta \rangle$ *is admissible, once* $R_Q^3 \subseteq \mathcal{I} \subseteq R_Q^2$.

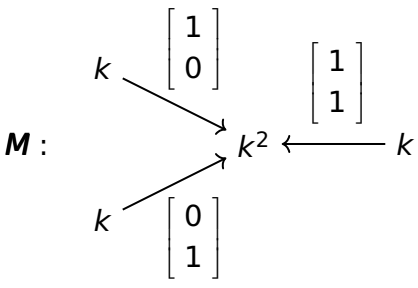
Similarly to what we have done with posets, we shall define quiver representations. Although depth is not intended on the topic, it is of great importance to the motivation of the forthcoming section.

**Definition 1.6.5.** *A representation of a quiver* $Q = (Q_0, Q_1, s, t)$ *over the field* $k$ *is a tuple* $\boldsymbol{M} = \big((M_i)_{i \in Q_0}; (\varphi_\alpha)_{\alpha \in Q_1}\big)$, *where* $M_i$ *is a* $k$*–vector space for all* $i \in Q_0$ *and* $\varphi_\alpha : M_{s(\alpha)} \longrightarrow M_{t(\alpha)}$ *is a* $k$*–linear transformation for all* $\alpha \in Q_1$.

**Example 1.6.6.** *Let Q be the following quiver:*




*Then,* $\boldsymbol{M}$ *is a representation of Q:*



Let $\boldsymbol{M} = \left((M_i)_{i\in Q_0}; (\varphi_\alpha)_{\alpha\in Q_1}\right)$ and $\boldsymbol{M'} = \left((M'_i)_{i\in Q_0}; (\varphi'_\alpha)_{\alpha\in Q_1}\right)$ be representations of a quiver $Q$. A *morphism of representations* $f : \boldsymbol{M} \longrightarrow \boldsymbol{M'}$ is a family of $k$–linear maps $\left(f_i : M_i \longrightarrow M'_i\right)_{i\in Q_0}$ such that for each arrow $\alpha : a \longrightarrow b$ the equality $\varphi'_\alpha f_a = f_b \varphi_\alpha$ holds.

The connection between studying representations of quivers and representations of posets comes from the fact that, given a poset $\mathcal{S}$, there exists a quiver $Q_{\mathcal{S}} = (Q_0, Q_1, s, t)$ such that every element of $\mathcal{S}$ corresponds to a point of $Q_0$ and every relation $i \prec j$ in $\mathcal{S}$ corresponds uniquely to an arrow $\alpha : i \longrightarrow j$ in $Q_1$. By means of this association, we may find a set $\mathcal{I}$ of $k$–linear admissible relations in $Q_{\mathcal{S}}$ in a way that $kQ_{\mathcal{S}}*/\mathcal{I} \cong k\mathcal{S}*$ [Fon23, Section 1.4]. Finally, there's an equivalence of categories $Q : \mathfrak{sp}_{\mathcal{S}} \longrightarrow \mathrm{mod}_{sp}\, k\mathcal{S}*$, where $\mathrm{mod}_{sp}\, k\mathcal{S}*$ is the full subcategory of $\mathrm{mod}\, k\mathcal{S}*$ which consists of socle projective modules – modules $M$ such that $\mathrm{soc}(M)$ is projective [Sim92, Lemma 5.1].

## 1.7. Stability

The geometric approach to classifying algebras relies, roughly speaking, on the ideas described below. Given a basic, connected and finite–dimensional $k$–algebra $A$, there exists an admissible ideal $\mathcal{I}$ of the path algebra $kQ_A$ such that $A \cong kQ_A/\mathcal{I}$ [ASS06, Theorem 3.7] . In other words, studying $A$ is somehow equivalent to studying the representations of its associated quiver.

A dimension vector $\boldsymbol{\alpha} \in \mathbb{Z}^{|Q_0|}$ is an integral vector with positive components. For a dimension vector $\boldsymbol{\alpha}$, it can be considered the affine variety

$$R_{\boldsymbol{\alpha}}(Q) \doteq \bigoplus_{a\in Q_1} \mathrm{Hom}\left(k^{\alpha_{s(a)}}, k^{\alpha_{t(a)}}\right),$$

which parametrizes all representations with $\boldsymbol{\alpha}$ as their dimension–vector. One may define the algebraic group

$$G_{\boldsymbol{\alpha}} \doteq \prod_{i\in Q_0} GL_{\alpha_i}(k)$$

that acts on $R_{\boldsymbol{\alpha}}(Q)$ through change of basis such that any two elements of $R_{\boldsymbol{\alpha}}(Q)$ are isomorphic if and only if they belong to the same orbit of $G_{\boldsymbol{\alpha}}$.

Through Mumford's invariant geometric theory [MFK94], King considered spaces in which points naturally correspond to the isoclasses of the representations, having examined *semi–stable representations*, a subset of $R_{\boldsymbol{\alpha}}(Q)$.

The idea of stability for arbitrary abelian categories was formulated by Rudakov [Rud97], having been established stability conditions for the representation theory of quivers. Later on, Futorny and Iusenko [FI19b] defined the notion of stability for poset representations, an additive but not abelian category.

Analogously, it was defined the *Poset Variety* $Fl_{\mathcal{S},\boldsymbol{\alpha}}$, a projective variety which parametrizes the representations of $\mathcal{S}$ with dimension vector $\boldsymbol{\alpha}$. The group $GL_{\alpha_0}(k)$ acts on $Fl_{\mathcal{S},\boldsymbol{\alpha}}$ through base change in such way that the orbits of this action correspond bijectively to the isomorphism classes of representations of $\mathcal{S}$ with dimension vector $\boldsymbol{\alpha}$ [Fon23, Section 1.4].

For further details on this variety, as well as several examples, the reader is referred to Iusenko and Fonseca's paper [FI19a]. Through this idea, stability in the geometrical sense correlates to stability of representations of posets [FI19b, Section 5].

Before effectively defining stability for poset representations, a detail must be addressed. Given a poset $\mathcal{S}$ and a representation $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$, a *subrepresentation* $\boldsymbol{W}$ of $\boldsymbol{V}$ is a system $(W_0; W_i)_{i\in\mathcal{S}}$ such that $W_0 \subseteq V_0$ and $W_i \subseteq V_i$ for every $i \in \mathcal{S}$. The quotient $\boldsymbol{V}/\boldsymbol{W}$, however, needn't belong to $\mathfrak{sp}_{\mathcal{S}}$, as seen below.

**Example 1.7.1.** *Let $\mathcal{S}_1 = \{1\}$, a poset consisting of a single element. If we consider the representation $\boldsymbol{V} = (k; k) \in \mathfrak{sp}_{\mathcal{S}_1}$ and the subrepresentation $\boldsymbol{W} = (k; 0)$ of $\boldsymbol{V}$, the quotient $\boldsymbol{V}/\boldsymbol{W} = (k/k; k/0) \cong (0; k)$ is not a representation of $\mathcal{S}_1$.*

This issue may be overcome by the restriction to a specific kind of subrepresentations.

**Definition 1.7.2.** *Let $\mathcal{S}$ be a poset and $\boldsymbol{V} = (V_0; V_i)_{i\in\mathcal{S}}$ a subspace representation of it. A subrepresentation $\boldsymbol{W}$ is said to be* proper *if there exists a proper subspace $K \subset V_0$ such that $\boldsymbol{W} = \boldsymbol{V}_K = (K; V_i \cap K)_{i\in\mathcal{S}}$.*

Using the aforementioned terminology, if $\boldsymbol{V}_K$ is a proper subrepresentation of $\mathcal{S}$, then we have that $\boldsymbol{V}/\boldsymbol{V}_K \in \mathfrak{sp}_{\mathcal{S}}$ [FI19b, Proposition A.1] and we may then proceed to this section's main definition.

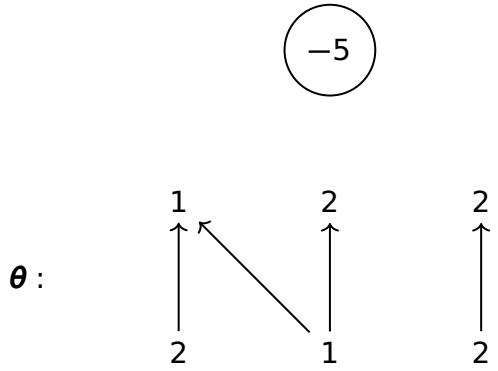
**Definition 1.7.3.** *A subspace representation* $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$ *is said to be* $\boldsymbol{\theta}$–stable (respectively $\boldsymbol{\theta}$–semistable) *if there exists a vector* $\boldsymbol{\theta} = (\theta_0; \theta_i)_{i \in S} \in \mathbb{Z}^{\mathcal{S}^*}$ *such that* $\boldsymbol{\theta} \cdot [\boldsymbol{dim}\, \boldsymbol{V}]^{tr} = 0$ *and* $\boldsymbol{\theta} \cdot [\boldsymbol{dim}\, \boldsymbol{V}_K]^{tr} < 0$ *(respectively* $\leq$*) for every proper subrepresentation* $\boldsymbol{V}_K$ *of* $\boldsymbol{V}$*.* $\boldsymbol{\theta}$ *is also referred to as a* weight*.*

The $\mu_{\boldsymbol{\theta}}$–slope of a representation $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$ is defined as

$$\mu_{\boldsymbol{\theta}}(\boldsymbol{V}) = \frac{1}{\dim V_0} \sum_{s \in \mathcal{S}} \theta_s \dim V_s$$

and $\boldsymbol{V}$ is said to be $\mu_{\boldsymbol{\theta}}$–stable (resp. semistable) if $\mu_{\boldsymbol{\theta}}(\boldsymbol{W}) < \mu_{\boldsymbol{\theta}}\boldsymbol{V})$ (resp. $\leq$) for every proper subrepresentation $\boldsymbol{W}$ of $\boldsymbol{V}$.

**Example 1.7.4.** *Consider* $\mathcal{S} = (N, 2)$ *from* Example 1.3.3 *and* $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$ *from* Example 1.5.2*. In this case,* $\boldsymbol{V}$ *is* $\boldsymbol{\theta}$*–stable, where* $\boldsymbol{\theta} \in \mathbb{Z}^{\mathcal{S}^*}$ *is given by:*




*We begin proving that* $\boldsymbol{V}$ *is indeed stable by observing that*

$$\boldsymbol{\theta} \cdot [\boldsymbol{dim}\, \boldsymbol{V}]^{tr} = (-5; 2, 1, 1, 2, 2, 2) \cdot (3; 1, 2, 1, 2, 1, 2) = 0.$$

*The task of checking that* $\boldsymbol{\theta} \cdot [\boldsymbol{dim}\, \boldsymbol{W}]^{tr} < 0$ *for every proper subrepresentation* $\boldsymbol{W}$ *of* $\boldsymbol{V}$ *may be accomplished by considering all maximal subdimension vectors [FI19b, Appendix C].*

$(1; 0, 0, 0, 0, 1, 1)$ $(1; 0, 0, 0, 1, 0, 1)$ $(1; 0, 1, 0, 0, 0, 1)$ $(1; 0, 1, 1, 1, 0, 0)$
$(1; 1, 1, 0, 0, 0, 0)$ $(2; 0, 1, 0, 1, 1, 2)$ $(2; 0, 1, 1, 1, 1, 1)$ $(2; 0, 1, 1, 2, 0, 1)$
$(2; 1, 1, 0, 1, 1, 1)$ $(2; 1, 2, 1, 1, 0, 1)$

Not only did Futorny and Iusenko define stability for the category of poset representations, but also they achieved a practical result when it comes to calculating the weight with which some representations are stable. The bilinear form $b_{\mathcal{S}} : \mathbb{Z}^{\mathcal{S}^*} \times \mathbb{Z}^{\mathcal{S}^*} \longrightarrow \mathbb{Z}$ such that $b_{\mathcal{S}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \boldsymbol{\alpha} \cdot C_{\mathcal{S}^*}^{-1} \cdot \boldsymbol{\beta}$ is a key factor in this calculation.

Given a dimension vector $\boldsymbol{\alpha} \in \mathbb{Z}^{\mathcal{S}}$, the *support* of $\boldsymbol{\alpha}$ is defined as the full subposet $\mathbf{supp}\,\boldsymbol{\alpha} = \{i \in \mathcal{S} \mid \boldsymbol{\alpha}(i) \neq 0\}$. An indecomposable representation $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$ is *sincere* if $\mathbf{supp}\,\mathbf{dim}\,\boldsymbol{V} = \mathcal{S}^*$. A poset $\mathcal{S}$ is called *sincere* if it has got at least one sincere representation. The set of sincere posets was given by Iusenko and Futorny [FI19b, Proposition 4.2].

**Proposition 1.7.5 ([FI19b], Proposition 4.3).** *Given $\mathcal{S}$ a sincere poset and $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$ a sincere representation, $\boldsymbol{V}$ is stable with a form*

$$\boldsymbol{\theta}(\boldsymbol{W}) = b_{\mathcal{S}}(\boldsymbol{dim}\,\boldsymbol{V}, \boldsymbol{dim}\,\boldsymbol{W}) - b_{\mathcal{S}}(\boldsymbol{dim}\,\boldsymbol{W}, \boldsymbol{dim}\,\boldsymbol{V})$$

*for any proper subrepresentation $\boldsymbol{W}$ of $\boldsymbol{V}$. It may be checked that the components of the form $\boldsymbol{\theta} \in \mathbb{Z}^{\mathcal{S}^*}$ are*

$$\theta_0 = -\sum_{\substack{t \in \mathcal{S}^* \\ t \to 0}} \alpha_t + \sum_{t \in \mathcal{S}} r(t, 0)\alpha_t$$
$$\theta_i = \sum_{\substack{t \in \mathcal{S}^* \\ i \to t}} \alpha_t - \sum_{\substack{t \in \mathcal{S}^* \\ t \to i}} \alpha_t - \sum_{t \in \mathcal{S}} r(i, t)\alpha_t,$$

*where $r(i, t)$ stands for the maximal number of k–linear independent minimal commutativity relations starting in $i$ and ending in $t$, and $\alpha_t = \dim V_t$ for $t \in \mathcal{S}^*$.*

Jordan–Hölder and Harder–Narasimhan filtrations with stable and semistable factors had been proven to exist by Rudakov [Rud97], for an abelian category, and by Hille and Peña, for representations of quivers [HdlP02]. Iusenko and Futorny proved the existence of these filtrations in the case of representations of posets, as we shall see below.

**Proposition 1.7.6 ([FI19b], Preposition 2.3).** *Given $\boldsymbol{V} = (V_0; V_s)_{s \in \mathcal{S}} \in \mathfrak{sp}_{\mathcal{S}}$, there exists a unique filtration of vector subspaces*

$$0 = K^0 \subset K^1 \subset \cdots \subset K^h = V_0$$

*inducing a filtration of* $\boldsymbol{V}$

$$0 = \boldsymbol{V}^0 \subset \boldsymbol{V}^1 \subset \cdots \subset \boldsymbol{V}^h = \boldsymbol{V},$$

*in which* $\boldsymbol{V}^i = \boldsymbol{V}_{K^i} = (K^i; V_s \cap K^i)_{s\in\mathcal{S}}$, *such that*

***i.*** $\boldsymbol{V}^i/\boldsymbol{V}^{i-1}$ *is* $\mu_\theta$*–semistable for every* $i = 1, \ldots, h$ *and*

***ii.*** $\mu_{\boldsymbol{\theta}}(\boldsymbol{V}^i/\boldsymbol{V}^{i-1}) > \mu_{\boldsymbol{\theta}}(\boldsymbol{V}^{i+1}/\boldsymbol{V}^i)$ *for every* $i = 1, \ldots, h-1$.

**Proposition 1.7.7 ([FI19b], Preposition 2.3).** *Given a* $\mu_{\boldsymbol{\theta}}$*–semistable representation* $\boldsymbol{V} = (V_0; V_s)_{s\in\mathcal{S}} \in \mathfrak{sp}_{\mathcal{S}}$, *there exists a filtration of vector subspaces*

$$0 = K^0 \subset K^1 \subset \cdots \subset K^h = V_0$$

*inducing a filtration of* $\boldsymbol{V}$

$$0 = \boldsymbol{V}^0 \subset \boldsymbol{V}^1 \subset \cdots \subset \boldsymbol{V}^h = \boldsymbol{V},$$

*in which* $\boldsymbol{V}^i = \boldsymbol{V}_{K^i} = (K^i; V_s \cap K^i)_{s\in\mathcal{S}}$, *such that*

***i.*** $\boldsymbol{V}^i/\boldsymbol{V}^{i-1}$ *is* $\mu_\theta$*–stable for every* $i = 1, \ldots, h$ *and*

***ii.*** $\mu_{\boldsymbol{\theta}}(\boldsymbol{V}^i/\boldsymbol{V}^{i-1}) = \mu_{\boldsymbol{\theta}}(\boldsymbol{V}^{i+1}/\boldsymbol{V}^i)$ *for every* $i = 1, \ldots, h-1$.

# Chapter 2

# A Maximal–Element Poset Differentiation

The algorithm to be presented in this section has been of immeasurable importance to the field of representation theory. Originally developed as a tool for solving the *Second Conjecture of Brauer–Thrall* [NR75], it was a key factor in classifying posets of finite representation type [Kle75].

## 2.1. The Algorithm of Nazarova and Roiter

Before properly describing the algorithm, we need to establish a rather simple concept. The *width* of a poset $\mathcal{S}$, denoted $\omega(\mathcal{S})$, is the size of the largest antichain in $\mathcal{S}$, i.e. the largest amount of pairwise incomparable elements of $\mathcal{S}$.

**Definition 2.1.1.** *Given a maximal element $c \in \mathcal{S}$ such that $\omega(\mathcal{S} \setminus c^{\nabla}) \leq 2$, where $c^{\nabla} = \{i \in \mathcal{S} \mid i \preceq c\}$, a new poset may be defined:*

$$\partial_c \mathcal{S} := (\mathcal{S} \setminus \{c\}) \cup \Omega,$$

*where $\Omega = \{r \vee s \mid r, s, c \text{ are incomparable in } \mathcal{S}\}$. The partial order relations in $\partial_c \mathcal{S}$ are given by*

***i.*** *keeping all relations between elements of $\mathcal{S}$ in $\partial_c \mathcal{S}$;*

***ii.*** *$t \preceq s \vee r \iff t \preceq s$ or $t \preceq r$;*

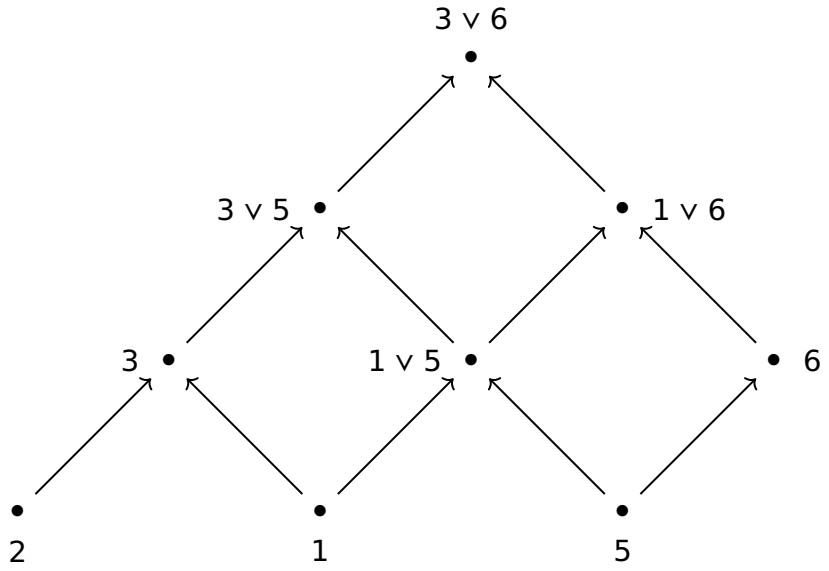
***iii.*** *$r \vee s \preceq t \iff r \preceq t$ and $s \preceq t$;*

***iv.*** *$r \vee s \preceq r' \vee s' \iff (r \preceq r'$ or $r \preceq s')$ and $(s \preceq r'$ or $s \preceq s')$.*

*We say that $\mathcal{S}$ is c–differentiable and $\partial_c S$ is called* a differential of $\mathcal{S}$ with respect to the maximal element *c.*

**Example 2.1.2.** *Let $\mathcal{S} = (N, 2)$ from* Example 1.3.3. *$\mathcal{S}$ is 4–differentiable and the differential of $\mathcal{S}$ with respect to the maximal element 4 is given by*

$\partial_4\mathcal{S}$ :




The theorem below is due to Nazarova and Roiter [NR75] and it was of great importance in the study of posets of finite representation–type. The reader is referred to Simson's textbook, where it can be found a sketch of proof using the original ideas ([Sim92], Section 4.2), as well as a more understandable proof using Gabriel's reduction functor ([Sim92], Section 6.2).

**Theorem 2.1.3 ([Sim92] Theorem 4.3).** *Given a finite c–differentiable poset $\mathcal{S}$ such that $\omega(S) \leq 3$, it follows that*

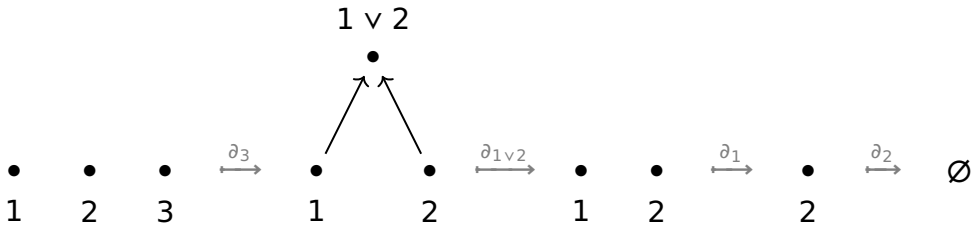
***i.*** *$\mathcal{S}$ and $\partial_c\mathcal{S}$ are of the same representation type, either* finite *or* infinite*;*

***ii.*** *$\#\operatorname{Ind}\mathbb{M}_{\partial_c\mathcal{S}} < \#\operatorname{Ind}\mathbb{M}_{\mathcal{S}}$, in case both numbers are finite.*

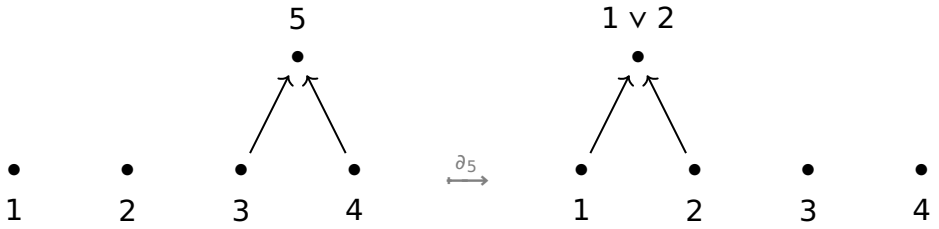
A pretty straightforward yet useful consequence of this theorem may be inferred. It consists of a practical apparatus for determining whether a given poset is of finite representation type or not.

**Corollary 2.1.4.** *A poset $\mathcal{S}$ is of finite representation type if and only if $\omega(\mathcal{S}) \leq 3$ and the maximal–element differentiation procedure reduces $\mathcal{S}$ to the empty poset in finitely many steps.*

**Example 2.1.5.** *Here we analyse the use of the previous corollary by successively applying Nazarova and Roiter's algorithm.*




*Having reached the empty poset, we may then conclude that the initial poset is of finite representation type. Now let's take a look at another case.*




*The element* 5 *is the only one we may differentiate the initial poset with respect to. The resulting poset after the procedure is isomorphic to the one we began with, meaning that it is of infinite representation type.*

## 2.2. A Maximal Element Differentiation Functor

Gabriel gave Nazarova and Roiter's algorithm a categorical interpretation ([Gab73]). Given a finite c–differentiable poset $\mathcal{S}$, there can be defined a functor $\boldsymbol{\partial}_c : \mathfrak{sp}_{\mathcal{S}} \longrightarrow \mathfrak{sp}_{\partial_c\mathcal{S}}$, where:

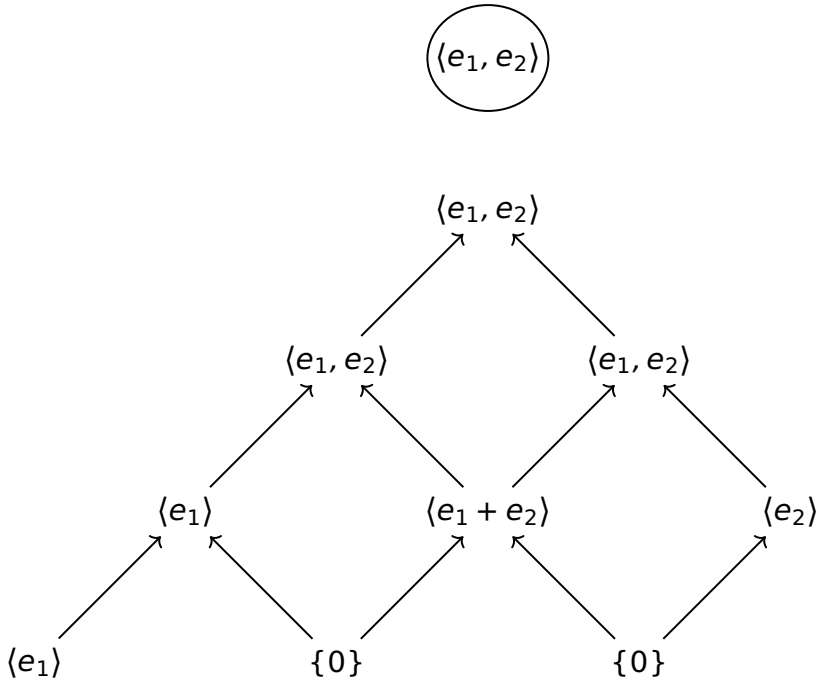
**i.** for each $\boldsymbol{V} = (V_0\,; V_i)_{i\in\mathcal{S}}$, the representation $\boldsymbol{\partial}_c(\boldsymbol{V}) = (V'_0\,; V'_j)_{j\in\partial_c\mathcal{S}}$ is given by:

$$\begin{aligned} V'_0 &= V_c && \\ V'_j &= V_c \cap V_j && \text{for } j \in \mathcal{S} \\ V'_{r\vee s} &= V_c \cap (V_r + V_s) && \text{for } r \vee s \in \Omega. \end{aligned}$$

**ii.** if $f : \boldsymbol{V} \longrightarrow \boldsymbol{W}$ is a morphism in $\mathfrak{sp}_{\mathcal{S}}$, then $\boldsymbol{\partial}_c(f) : \boldsymbol{\partial}_c(\boldsymbol{V}) \longrightarrow \boldsymbol{\partial}_c(\boldsymbol{W})$ is naturally given by restricting $f$ to $V_c$.

**Example 2.2.1.** *Consider* $\mathcal{S} = (N, 2)$ *from* Example 1.3.3 *and* $\partial_4\mathcal{S}$*, its differential poset with respect to the maximal element* 4 *shown in* Example 2.1.2*. Applying Gabriel's functor to the representation* $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$ *from* Example 1.5.2*, we get the differential representation of* $\boldsymbol{V}$ *with respect to the maximal element* 4*:*

$\boldsymbol{\partial_4(V)}$ :




In order to avoid confusion, it is important to point out that, by $\partial_c\mathcal{S}*$, we mean the poset $\partial_c\mathcal{S}$ extended by a maximal element, that is $(\partial_c\mathcal{S})*$, and *not* $\partial_c(\mathcal{S}*)$.

Fonseca's work ([Fon23], Section 3.2) showed that there exists a linear transformation connecting a subspace representation of a poset to its differential representation provided that a certain condition is satisfied. A subspace representation $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$ is said to be *generic* if the intersection $\bigcap_{j\in J}\left(\sum_{i\in I_j} V_i\right)$ for any $I_j \subseteq \mathcal{S}$ and $J \subset \mathbb{N}$ is the smallest possible for the dimension vector $\mathbf{dim}\,\boldsymbol{V}$.

If $\mathcal{S}$ is $c$–differentiable and $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$ is a generic representation, we shall see that there exists a transformation $D_{\boldsymbol{V}} : \mathbb{Z}^{\mathcal{S}*} \longrightarrow \mathbb{Z}^{\partial_c\mathcal{S}*}$ such that $\mathbf{dim}\,\boldsymbol{\partial}_c(\boldsymbol{V}) = D_{\boldsymbol{V}}(\mathbf{dim}\,\boldsymbol{V})$. Fixing bases for the vector spaces $V_0$ and $V_c$, the transformation may be equivalently written as a matrix $D_{\boldsymbol{V}} \in \mathbb{M}_{\mathcal{S}*\times\partial_c\mathcal{S}*}(\mathbb{Z})$, meaning that $\mathbf{dim}\,\boldsymbol{\partial}_c(\boldsymbol{V}) = \mathbf{dim}\,\boldsymbol{V} \cdot D_{\boldsymbol{V}}$. Avoiding cumbersome notation, we write $\boldsymbol{\alpha} = \mathbf{dim}\,\boldsymbol{V}$, $\boldsymbol{V'} = \boldsymbol{\partial}_c(\boldsymbol{V})$ and $\boldsymbol{\alpha'} = \mathbf{dim}\,\boldsymbol{\partial}_c(\boldsymbol{V})$. Since $\boldsymbol{V}$ is generic, the following equalities hold:

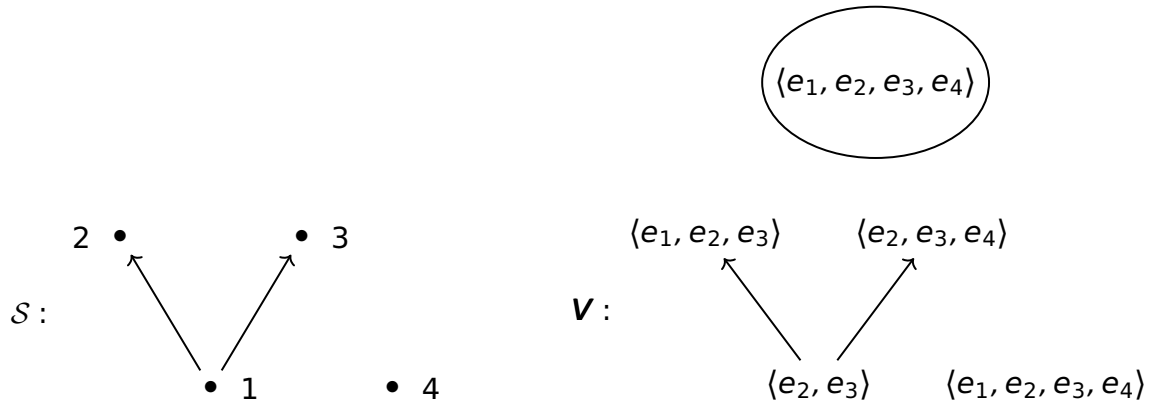
$$\begin{aligned}\alpha'_m &= \dim\left(V_m \cap V_c\right)\\ &= \max\left(\alpha_c + \alpha_m - \alpha_0\ ,\ \dim \sum_{j \prec c,m} V_j\right) \text{ for all } m \in \mathcal{S} \setminus \alpha^{\triangledown};\end{aligned}$$

$$\begin{aligned}\alpha'_{r\vee s} &= \dim\left((V_r + V_s) \cap V_c\right)\\ &= \max\left(\min\left(\dim \sum_{\substack{r\prec x\\ s\prec x}} V_x\ ,\ \alpha_r + \alpha_s - \dim \sum_{\substack{y\prec r\\ y\prec s}} V_y\right) + \alpha_c - \alpha_0\ ,\ \dim \sum_{\substack{z\prec a\\ z\prec r\vee s}} V_z\right)\end{aligned}$$

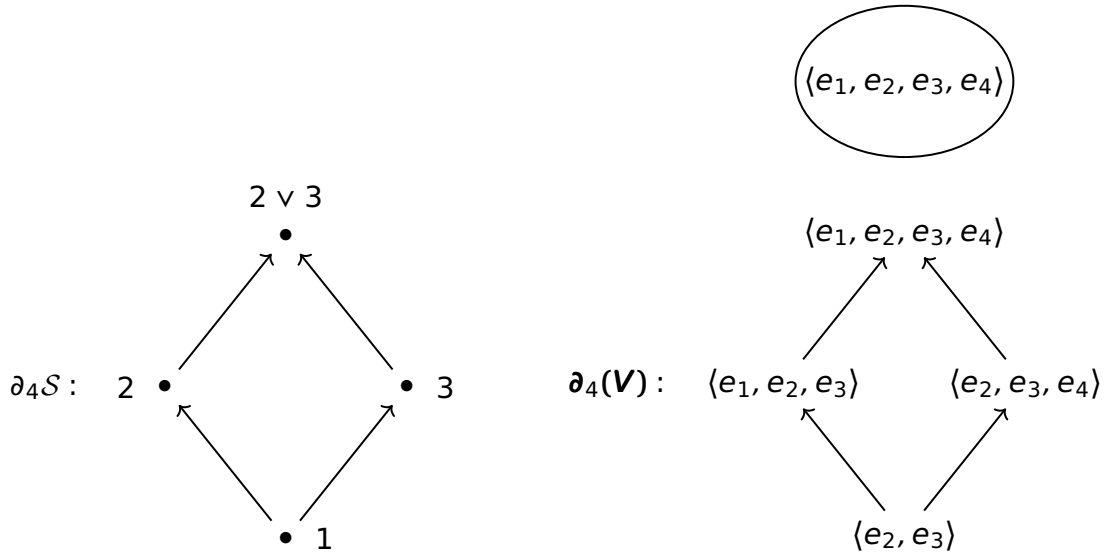
for all $r \vee s \in \Omega$.

Being $\boldsymbol{V}$ a generic representation, for every subposet $U \subseteq \mathcal{S}$, the dimension of $\sum_{u\in U} V_u$ may be written in terms of a linear combination of the vector spaces $V_s$, where $s \in u^{\triangledown}$ and $u \in U$. This way, may be built the aforementioned matrix $D_{\boldsymbol{V}} \in \mathbb{M}_{\mathcal{S}*\times\partial_c\mathcal{S}*}(\mathbb{Z})$ such that $\mathbf{dim}\,\boldsymbol{\partial}_c(\boldsymbol{V}) = \mathbf{dim}\,\boldsymbol{V} \cdot D_{\boldsymbol{V}}$.

**Example 2.2.2.** *Consider the poset $\mathcal{S}$ and the representation $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$ depicted by the following diagrams.*




*Then $\mathcal{S}$ is 4–differentiable and the differential poset $\partial_4\mathcal{S}$, as well as the differential representation $\boldsymbol{\partial}_4(\boldsymbol{V}) = (V'_0; V'_i)_{i\in\partial_4\mathcal{S}}$ are, respectively,*




*One notices that $V_m + V_4 = V_0$ for every $m \in \mathcal{S} \setminus a^{\nabla}$, which allows us to use Grassman's Intersection Formula ([Car17], Theorem 6.21) in a pretty convenient way:*

$$\begin{aligned}
\dim V'_1 &= \dim V_1 \cap V_4 = \dim V_1 + \dim V_4 - \dim V_0 = 2 + 4 - 4 = 2\\
\dim V'_2 &= \dim V_2 \cap V_4 = \dim V_2 + \dim V_4 - \dim V_0 = 3 + 4 - 4 = 3\\
\dim V'_3 &= \dim V_3 \cap V_4 = \dim V_3 + \dim V_4 - \dim V_0 = 3 + 4 - 4 = 3 \textit{ and}\\
\dim V'_{2\vee 3} &= \dim (V_2 + V_3) \cap V_4 = \dim V_2 + \dim V_3 - \dim V_1 + \dim V_4 - \dim V_0\\
&= 3 + 3 - 2 + 4 - 4 = 4.
\end{aligned}$$

*In other words, we may write the following matrix $D_{\boldsymbol{V}} \in \mathbb{M}_{5\times 5}(\mathbb{Z})$:*

$$D_{\boldsymbol{V}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 2\vee 3 \end{array}\\ \begin{bmatrix} 0 & -1 & -1 & -1 & -1\\ 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

*which leads to $\boldsymbol{\dim \partial_c(\boldsymbol{V}) = \dim \boldsymbol{V} \cdot D_{\boldsymbol{V}}}$.*

## 2.3. On the Congruence of $C_{\mathcal{S}}^{-1}$ and $C_{\partial_c\mathcal{S}}^{-1}$

Fonseca has also shown that, given a finite poset $\mathcal{S}$, its inverse Cartan Matrix $C_{\mathcal{S}}^{-1}$ and the inverse Cartan Matrix of its differential with respect to a maximal element $C_{\partial_c\mathcal{S}}^{-1}$ are congruent under some conditions [Fon23]. We begin with an example to illustrate this statement, followed by the theorem that proves this equality and gives us a practical technique to build this congruence.

**Example 2.3.1.** *Consider $\mathcal{S} = (N, 2)$, the poset from* Example 1.3.3. *We've seen that $\mathcal{S}$ is 4–differentiable and its differential $\partial_4\mathcal{S}$ was given in* Example 3.1.3. *The Cartan matrices of $\mathcal{S}^*$ and $\partial_4\mathcal{S}^*$ are, respectively,*

$$C_{\mathcal{S}^*} = \begin{array}{c|ccccccc} & 0 & 1 & 3 & 5 & 6 & 2 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 2 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ \hline 4 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \quad \textit{and}$$

$$C_{\partial_4\mathcal{S}^*} = \begin{array}{c|ccccccccc} & 0 & 1 & 3 & 5 & 6 & 2 & 1\vee5 & 3\vee5 & 1\vee6 & 3\vee6 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 5 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 6 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ \hline 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1\vee5 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 3\vee5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1\vee6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 3\vee6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}.$$

*The choice of ordering the elements this particular way and writing the matrices as blocks will be clear when the section's main theorem is stated. The inverse Cartan matrices are*

$$C_{\mathcal{S}}^{-1}{}_{*} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$C_{\partial_4\mathcal{S}}^{-1}{}_{*} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

*A plain calculation shows that* $C_{\mathcal{S}}^{-1}{}_{*} = D_{\mathcal{S}} \cdot C_{\partial_4\mathcal{S}}^{-1}{}_{*} \cdot D_{\mathcal{S}}^{tr}$, *where*

$$D_{\mathcal{S}} = \begin{array}{c|c|ccccc|cccc} & & 0 & 1 & 3 & 5 & 6 & 2 & 1\vee5 & 3\vee5 & 1\vee6 & 3\vee6 \\ \hline 0 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 5 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ \hline 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 4 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}.$$

The following auxiliary matrices play important role in the upcoming theorem. Let $A$ and $B$ be subposets of $\mathcal{S}$.

**i.** The Cartan Matrix of $A$ and $B$, denoted $C_{AB} \in \mathbb{M}_{|A|\times|B|}(\mathbb{Z})$, is given by

$$(C_{AB})_{ij,\, i\in A,\, j\in B} = \begin{cases} 1\,, \text{ if } i \preceq j\,; \\ 0, \text{ otherwise.} \end{cases}$$

**ii.** The matrix $E_A \in \mathbb{M}_{|A|\times 1}(\mathbb{Z})$ consists of 1 in all of its entries:

$$E_A = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

**iii.** The matrix $0_{A\times B} \in \mathbb{M}_{|A|\times|B|}(\mathbb{Z})$ consists of 0 in all of its entries:

$$\mathbf{0}_{A\times B} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.$$

**Theorem 2.3.2 ([Fon23] Theorem 3.3.1).** *Let $\mathcal{S}$ be a c–differentiable poset and consider $M = \mathcal{S} \setminus c^{\triangledown}$ and $N = c^{\triangledown}$ subposets of $\mathcal{S}$. If $M$ has got no global minimum, i.e. there is no $s \in \mathcal{S}$ such that $s \preceq m$ for all $m \in M$, then*

$$C_{\mathcal{S}*}^{-1} = D_{\mathcal{S}} \cdot C_{\partial_c\mathcal{S}*}^{-1} \cdot D_{\mathcal{S}}^{tr}$$

*where $D_{\mathcal{S}} \in \mathbb{M}_{|\mathcal{S}*|\times|\partial_c\mathcal{S}*|}(\mathbb{Z})$ is given by*

$$D_{\mathcal{S}} = \begin{array}{c} \\ \{0\} \\ M \\ \{c\} \\ N \end{array} \begin{array}{c} \begin{array}{cccc} \{c\} & M & N & \Omega \end{array} \\ \begin{bmatrix} 0 & -E_M^{tr} & 0_{1\times N} & -E_\Omega^{tr} \\ 0_{M\times 1} & \mathrm{Id}_M & 0_{M\times N} & C_M^{-1} C_{M\Omega} \\ 1 & E_M^{tr} & 0_{1\times N} & E_\Omega^{tr} \\ 0_{N\times 1} & 0_{N\times M} & \mathrm{Id}_N & C_N^{-1}(C_{N\Omega} - C_{NM} C_M^{-1} C_{M\Omega}) \end{bmatrix} \end{array}.$$

*Sketch of Proof.* Having fixed the decompositions $\mathcal{S}* = \{0\} \cup M \cup \{c\} \cup N$ and $\partial_c\mathcal{S}* = \{0\} \cup M \cup N \cup \Omega$, both $C_{\mathcal{S}*}$ and $C_{\partial_c\mathcal{S}*}$ may be written as block matrices:

$$C_{\mathcal{S}}* = \begin{array}{c} \\ \{0\} \\ M \\ N \\ \{a\} \end{array} \begin{array}{c} \begin{array}{cccc} \{0\} & M & N & \{a\} \end{array} \\ \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} & 0 \\ E_M & C_M & \mathbf{0} & \mathbf{0} \\ E_N & C_{NM} & C_N & E_N \\ 1 & \mathbf{0} & \mathbf{0} & 1 \end{bmatrix} \end{array} \text{ and}$$

$$C_{\partial_c\mathcal{S}}* = \begin{array}{c} \\ \{0\} \\ M \\ N \\ \Omega \end{array} \begin{array}{c} \begin{array}{cccc} \{0\} & M & N & \Omega \end{array} \\ \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ E_M & C_M & \mathbf{0} & C_{M\Omega} \\ E_N & C_{NM} & C_N & C_{N\Omega} \\ E_\Omega & C_{\Omega M} & \mathbf{0} & C_\Omega \end{bmatrix} \end{array}.$$

Writing Cartan matrices as block matrices allows us to have their inverse matrices in a rather general form:

$$C_{\partial_c\mathcal{S}}* = \begin{array}{c} \\ \{0\} \\ M \\ N \\ \Omega \end{array} \begin{array}{c} \begin{array}{cccc} \{0\} & M & N & \Omega \end{array} \\ \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Delta_1(C_{M\Omega}C_\Omega^{-1}E_\Omega - E_M) & \Delta_1 & \mathbf{0} & -C_M^{-1}C_{M\Omega}\Delta_2 \\ * & C_N^{-1}(C_{N\Omega}C_\Omega^{-1}C_{\Omega M} - C_{NM})\Delta_1 & C_N^{-1} & C_N^{-1}(C_{NM}C_M^{-1}C_{M\Omega} - C_{N\Omega})\Delta_2 \\ * * & -C_\Omega^{-1}C_{\Omega M}\Delta_1 & \mathbf{0} & \Delta_2 \end{bmatrix} \end{array},$$

where

$$\Delta_1 = (C_M - C_{M\Omega}C_\Omega^{-1}C_{\Omega M})^{-1},$$

$$\Delta_2 = (C_\Omega - C_{\Omega M}C_M^{-1}C_{M\Omega})^{-}1,$$

$$* = C_N^{-1}[-E_n + C_{N\Omega}C_\Omega^{-1}E_\Omega - (C_{NM} - C_{N\Omega}C_\Omega^{-1}C_{\Omega M})\Delta_1(C_{M\Omega}C_\Omega^{-1}E_\Omega - E_M)] \text{ and}$$

$$* * = -C_\Omega^{-1}[E_\Omega + C_{\Omega M}\Delta_1(C_{M\Omega}C_\Omega^{-1}E_\Omega - E_m)].$$

The equality $C_{\mathcal{S}*}^{-1} = D_{\mathcal{S}} \cdot C_{\partial_c\mathcal{S}*}^{-1} \cdot D_{\mathcal{S}}^{tr}$ holds if and olny if the following equations are satisfied:

$$E_M^{tr} C_M^{-1} E_M + T(C_{\Omega M} C_M^{-1} E_M - E_\Omega) = 1$$

$$-E_M^{tr} C_M^{-1} + T(C_{M\Omega}^{tr} C_M^{-tr} - C_{\Omega M} C_M^{-1}) = 0$$

$$T(C_{N\Omega} - C_{NM} C_M^{-1} C_{M\Omega}) = 0,$$

where $T = E_M^{tr} C_M^{-1} C_{M\Omega} \Delta_2 - E_\Omega^{tr} \Delta_2$.

One shows that the case for $T = (1, 0, \ldots, 0)$ is valid, proceeding with induction on the number of elements of $M$ to prove that, indeed, $T = (1, 0, \ldots, 0)$.

□

## 2.4. Stability after a maximal–element differentiation

One of Fonseca's main results explores the behaviour of stable representations after Nazarova & Roiter's maximal–element differentiation procedure. In order to do so, an integration map plays a crucial role and is presented henceforth.

We've seen in Theorem 2.3.2 that it is quite convenient to consider a poset $\mathcal{S}$ as a disjoint union of subposets $M$ and $N$. One checks that $\omega(M) \leq 2$, which allows us to conclude that there exist at most two maximal disjoint and totally ordered suposets $M_I$ and $M_{II}$ such that $M = M_I \cup M_{II}$ [Fon23, Remark 1.5.3]. This way, every element $w = r \vee s \in \Omega$ may be considered being $r \in M_I$ and $s \in M_{II}$.

An integration map was thought as being a procedure such that, given a $c$–differentiable poset $\mathcal{S}$ and a representation $\boldsymbol{V}' \in \mathfrak{sp}_{\partial_c\mathcal{S}}$, there could be constructed a representation $\int_c \boldsymbol{V}'$ with $\boldsymbol{\partial}_c(\int_c \boldsymbol{V}') = \boldsymbol{V}'$. As we shall see, the map depends on choices, meaning that it is not a functor nor unique.

**Definition 2.4.1 ([Fon23], Definition 1.5.7).** *Let $\mathcal{S}$ be a $c$–differentiable poset and $\boldsymbol{V}' \in \mathfrak{sp}_{\partial_c\mathcal{S}}$ a representation. For every $w \in \Omega$, we define*

- $X^I_w(\boldsymbol{V'}) = \dfrac{V'_w}{\sum_{u \preceq w} V'_u}$

- *a section* $\iota^{\boldsymbol{V'}}_w : X^I_w(\boldsymbol{V'}) \longrightarrow V'_w$ *of the canonical projection* $\pi_w$

- $X^{II}_w(\boldsymbol{V'}) = \mathrm{Im}\left(\iota^{\boldsymbol{V'}}_w + \mathrm{Id}_{X^I_w(\boldsymbol{V'})}\right)$

- $\overline{V_0} = \bigoplus_{u \in \Omega} X^I_u(\boldsymbol{V'})$

*and for every* $m \in M$, *we set* $\Omega^m = \{r \vee s \in \Omega \mid r \preceq m \text{ or } s \preceq m\}$.

*This way, the integral of* $\boldsymbol{V'}$ *is defined as the representation* $\int_c \boldsymbol{V'} = \boldsymbol{V} = (V_0; (V_m)_{m \in M}, V_c, (V_n)_{n \in N})$, *where*

$$V_0 = V'_0 \oplus \overline{V_0}$$

$$V_m = \begin{cases} V'_m + \sum_{w \in \Omega^m} \left(X^I_w(\boldsymbol{V'})\right) \text{ if } m \in M_I \\ V'_m + \sum_{w \in \Omega^m} \left(X^{II}_w(\boldsymbol{V'})\right) \text{ if } m \in M_{II} \end{cases}$$

$$V_n = V'_n \text{ if } n \in N.$$

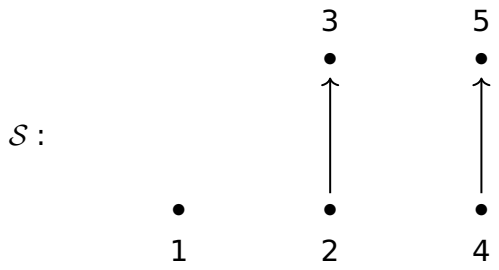
Fonseca's work presents a linear transformation $I_{\boldsymbol{V'}}$ that relates the dimension vectors of a representation $\boldsymbol{V'} \in \mathfrak{sp}_{\partial_c \mathcal{S}}$ and its integral $\int_c \boldsymbol{V'} \in \mathfrak{sp}_{\mathcal{S}}$. This result, however, works for a specific kind of representation, as seen below.

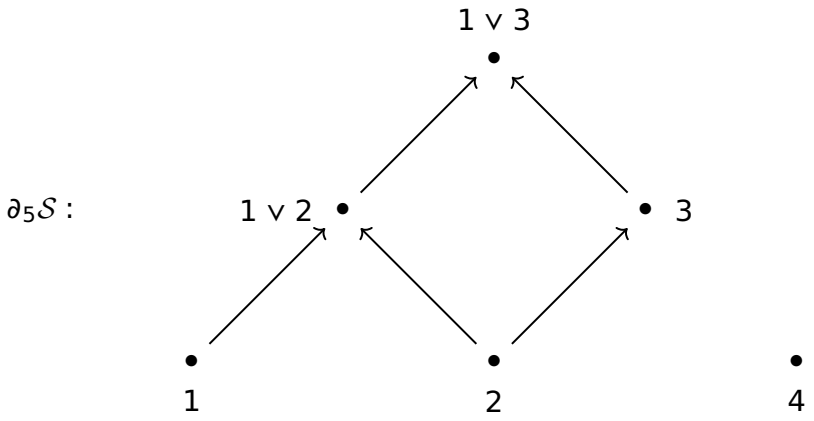
Given an element $w \in \Omega$, there will always exist two maximal elements $w_1, w_2 \in \{u \prec w \mid u \in \partial_c \mathcal{S}\}$ [Fon23, Remark 4.1.2], which means that

$$X^I_w(\boldsymbol{V'}) = \frac{V'_w}{\sum_{u \preceq w} V'_u} = \frac{V'_w}{V'_{w_1} + V'_{w_2}}.$$

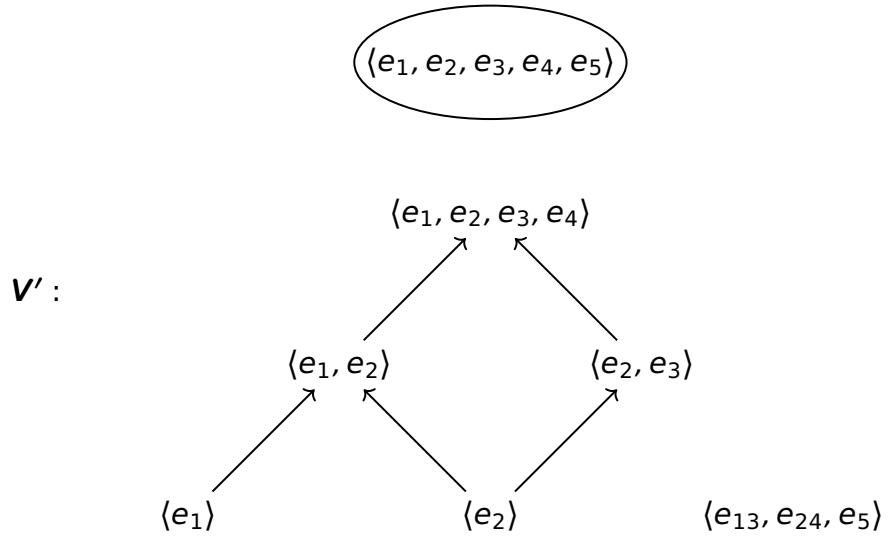
The representation $\boldsymbol{V'}$ is said to be *almost generic* if, for the dimension vector $\mathbf{dim}\,\boldsymbol{V'}$, the intersection $V'_{w_1} \cap V'_{w_2}$ is the smallest possible . Being $\boldsymbol{V'}$ almost generic and observed its dimension vector, it is possible to define a linear transformation $I_{\boldsymbol{V'}}$ such that $\mathbf{dim}\,\boldsymbol{V'} \cdot I_{\boldsymbol{V'}} = \mathbf{dim} \int_c \boldsymbol{V'}$ [Fon23, Definition 4.1.5].

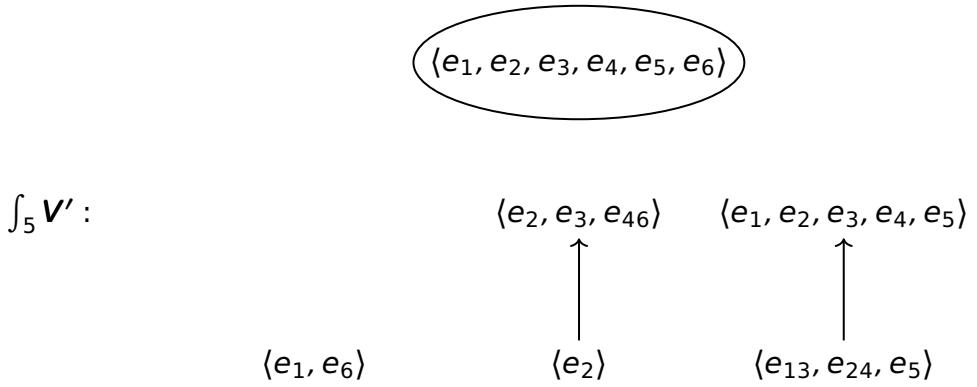
**Example 2.4.2.** *Consider the following poset $\mathcal{S}$ and its differential $\partial_5\mathcal{S}$*




*Given the following $\boldsymbol{V'} \in \mathfrak{sp}_{\partial_5\mathcal{S}}$, we may use Definition 2.4.1 to calculate $\int_5 \boldsymbol{V'}$:*





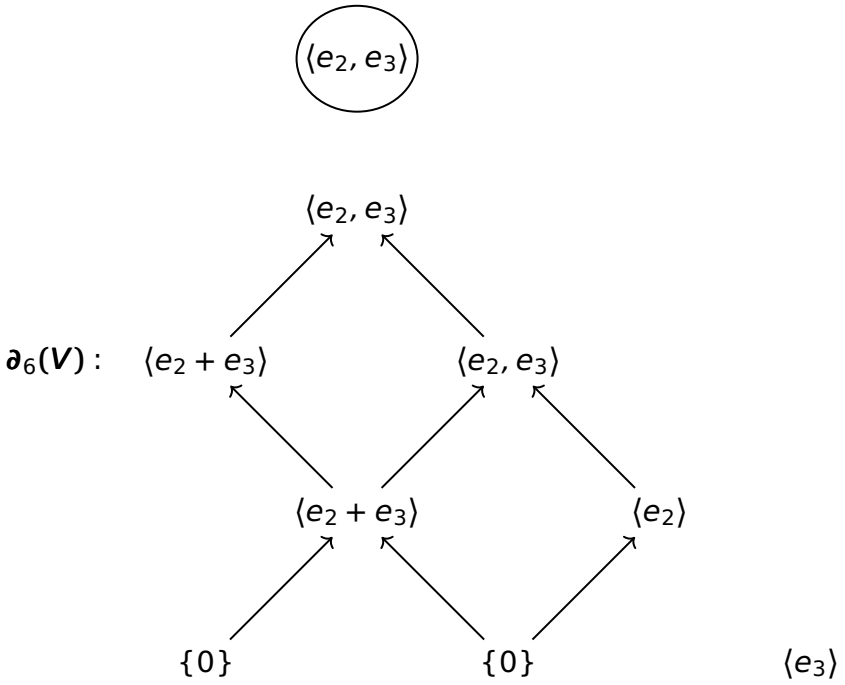

*The linear transformation $I_{V'}$ which yields* $\dim V' \cdot I_{V'} = \dim \int_5 V'$ *may be depicted by the matrix*

$$I_{V'} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 1 \vee 2 \\ 1 \vee 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}.$$

This section's main theorem was also stated by Fonseca and it describes the way an almost generic representation behaves under the maximal–element differentiation functor. The statement deals with positive stability, i.e. $\theta$–stable representations in which the last $|\mathcal{S}|$ coordinates of $\theta$ are positive. In this case, the representation is called *positively stable*.

**Theorem 2.4.3 ([Fon23], Theorem 4.2.4).** *If $\mathcal{S}$ is a c–differentiable poset, $V \in \mathfrak{sp}_{\mathcal{S}}$ is $\theta$–positively stable and $\partial_c(V)$ is almost generic, then $\partial_c(V)$ is $\theta'$–stable, where $\theta' = \theta \cdot I^{tr}_{\partial_c(V)}$.*

**Example 2.4.4.** *Consider the poset $\mathcal{S} = (N, 2)$ from Example 1.3.3 and the representation $V \in \mathfrak{sp}_{\mathcal{S}}$ from Example 1.5.2. $\mathcal{S}$ is 6–differentiable and $\partial_6(V)$ is given by*




*We've seen that* $\boldsymbol{V}$ *is* $\boldsymbol{\theta}$*–positively stable in Example 1.7.4. Applying Theorem 2.4.3, we get that* $\boldsymbol{\partial}_6(\boldsymbol{V})$ *is* $\boldsymbol{\theta}'$*–stable, where* $\boldsymbol{\theta}' = \boldsymbol{\theta} \cdot I_{\boldsymbol{V}'} = (-3; 1, 0, 2, 1, 2, 1, 0, 0)$*. For further details on the construction of* $I_{\boldsymbol{V}'}$*, as well as the calculations, the reader is referred to Fonseca's PhD thesis [Fon23, Example 4.2.5].*

# Chapter 3

# A Suitable–Pair Poset Differentiation

In the late 1970s, Zavadskij developed a new differentiation algorithm for posets which was, in his own words, applicable in a more general situation [Zav77]. Zavadskij and Nazarova were able to achieve a criterion for classificarion of posets of finite growth representation type, for instance, through this latter algorithm [ZN82]. Similarly to Fonseca's results on Nazarova and Roiter's maximal–element differentiation, this work intended to accomplish interesting results based on Zavadkij's algorithm.

## 3.1. The Algorithm of Zavadskij

Instead of differentiating with respect to a maximal element, like Nazarova and Roiter had done, Zavadskij proposed a procedure involving a pair of points. Not every pair of points is, however, eligible for the procedure, in the sense that a certain condition needs to be satisfied.

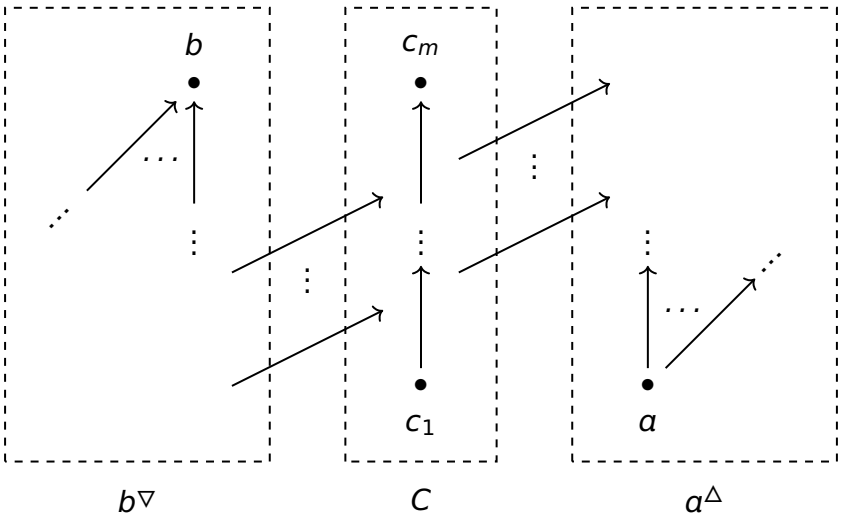
**Definition 3.1.1.** *A pair* $(a, b)$ *of elements of a poset* $\mathcal{S}$ *is called* suitable *if* $a \not\preceq b$ *and the subposet* $\mathcal{S}_b^a$ *of* $\mathcal{S}$ *is either empty or linearly ordered, where*

$$a^{\triangle} := \{i \in \mathcal{S} \mid a \preceq i\},$$

$$b^{\triangledown} := \{j \in \mathcal{S} \mid j \preceq b\} \text{ and}$$

$$\mathcal{S}_b^a := \mathcal{S} \setminus (a^{\triangle} \cup b^{\triangledown}).$$

Given $(a, b)$ a suitable pair, we may write $\mathcal{S} = b^{\triangledown} \cup C \cup a^{\triangle}$, where $C = \mathcal{S}_b^a$ is either $\{c_1, \ldots, c_m\}$ or empty; hence, $\mathcal{S}$ must have the form




**Definition 3.1.2.** *Given* $(a, b)$ *a suitable pair of* $\mathcal{S}$ *and* $C = \mathcal{S}_b^a = \{c_1, \ldots, c_m\}$, *we may define a new poset*

$$\partial_{(a,b)}\mathcal{S} := b^\nabla \cup C^- \cup C^+ \cup a^\Delta, \tag{3.1}$$

*where* $C^- = C^+ = \emptyset$ *if* $C = \emptyset$ *and* $C^- = \{c_1^- \prec \ldots \prec c_m^-\}$, $C^+ = \{c_1^+ \prec \ldots \prec c_m^+\}$ *otherwise. The partial order relation* $\preceq$ *in* $\partial_{(a,b)}\mathcal{S}$ *is defined as follows:*

***i.*** *all relations in* $\mathcal{S}$ *are kept between elements of* $a^\Delta + b^\nabla$;

***ii.*** $c_i^- \prec c_i^+$ *for all* $i = 1, \ldots, m$;

***iii.*** *if* $t \prec_{\mathcal{S}} b$, *then*

$$t \prec c_j^+ \iff t \prec_{\mathcal{S}} c_j \text{ and}$$

$$t \prec c_j^- \iff t \prec_{\mathcal{S}} c_j;$$

***iv.*** *if* $a \prec_{\mathcal{S}} s$, *then*

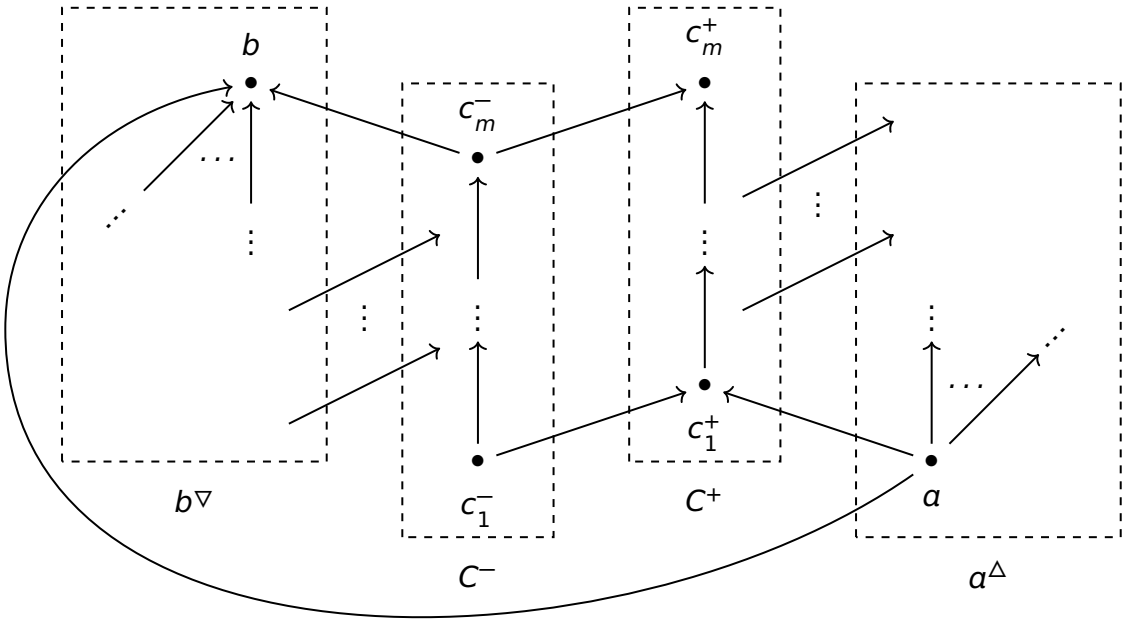
$$c_j^- \prec s \iff c_j \prec_{\mathcal{S}} s \text{ and}$$

$$c_j^+ \prec s \iff c_j \prec_{\mathcal{S}} s;$$

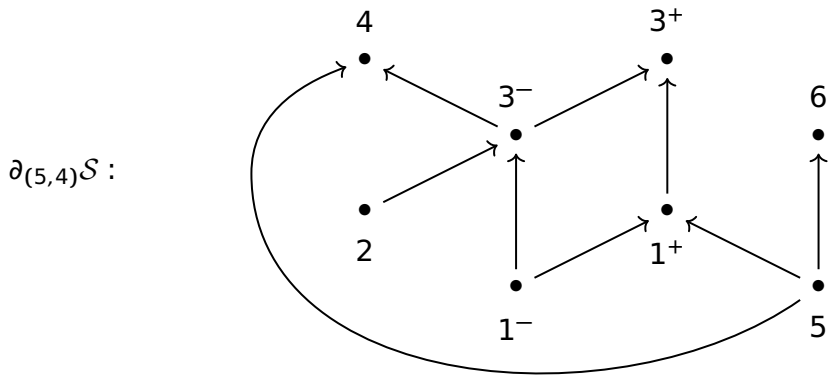
***v.*** *the relations* $a \prec b$, $a \prec c_1^+$ *and* $c_m^- \prec b$ *are added;*

***vi.*** *if* $s \prec t$ *and* $t \prec s$ *in* $\partial_{(a,b)}\mathcal{S}$, *then* $s$ *and* $t$ *are identified.*

The poset $\partial_{(a,b)}\mathcal{S}$ is called *a differential of* $\mathcal{S}$ *with respect to the suitable pair* $(a, b)$ and it has the following form:




**Example 3.1.3.** *Let* $\mathcal{S} = (N, 2)$ *from* Example 1.3.3*. The pair* $(5, 4)$ *is suitable and the differential of* $\mathcal{S}$ *with respect to it is given by:*




## 3.2. A Suitable–Pair Differentiation Map

Zavadskij himself gave a categorical interpretation of his differentiation algorithm through a map between representations of a poset $\mathcal{S}$ and its differential poset $\partial_{(a,b)}\mathcal{S}$.

**Definition 3.2.1.** *Given* $(a, b)$ *a suitable pair of* $\mathcal{S}$*, there can be defined a map* $\boldsymbol{\partial}_{(a,b)} : \mathfrak{sp}_{\mathcal{S}} \dashrightarrow \mathfrak{sp}_{\partial_{(a,b)}\mathcal{S}}$ *on isoclasses of objects as follows:*

***i.*** *Let* $C = \mathcal{S}_b^a = \{c_1 \prec \ldots \prec c_m\}$ *and* $\boldsymbol{V} = (V_0\,; V_j)_{j\in\mathcal{S}} \in \mathfrak{sp}_{\mathcal{S}}$.

***ii.*** *Choose* $U \subseteq V_a$ *a* $k$*–subspace such that* $V_a + V_b = U \oplus V_b$.

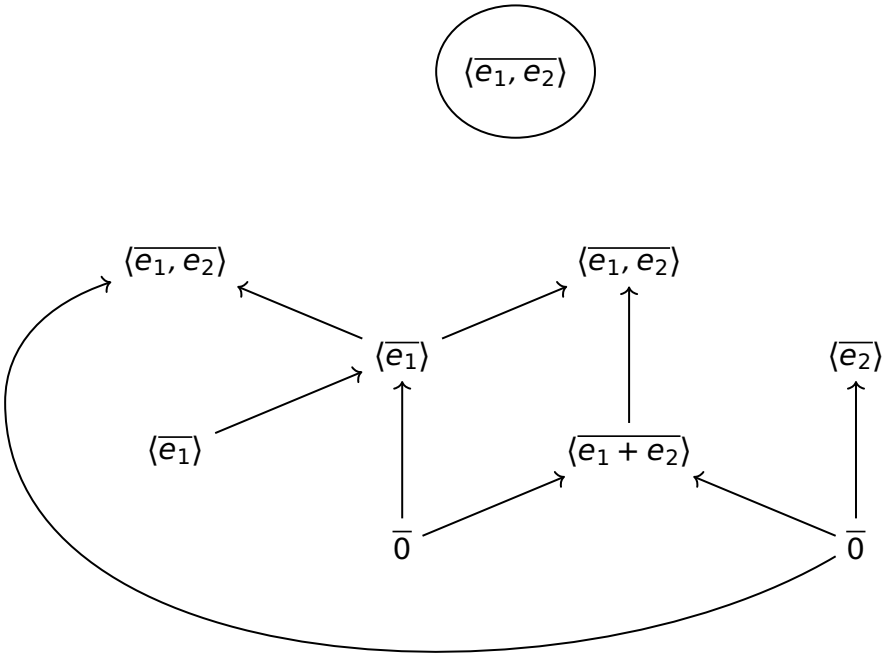
***iii.*** *Set* $\boldsymbol{\partial}_{(a,b)}(\boldsymbol{V}) := \boldsymbol{W} = (W_0\,; W_i)_{i\in\partial_{(a,b)}\mathcal{S}}$, *where*

$$\begin{aligned} W_0 &= V_0/U, \\ W_{c_j^-} &= \big((V_b \cap V_{c_j}) + U\big)/U && \text{for } j = 1,\ldots,m, \\ W_{c_j^+} &= (V_a + V_{c_j})/U && \text{for } j = 1,\ldots,m. \\ W_t &= (V_t + U)/U && \text{for } t \in a^{\triangle} + b^{\triangledown}. \end{aligned}$$

$\boldsymbol{\partial}_{(a,b)}$ *is called a* differentiation map with respect to the suitable pair $(a, b)$.

**Example 3.2.2.** *Let* $\mathcal{S} = (N, 2)$ *from* Example 1.3.3 *and consider* $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$ *from* Example 1.5.2. *We first observe that the sum* $V_5 + V_4$ *is direct, allowing us to choose* $U = V_5$, *which satisfies the condition* ***ii.*** *from* Definition 3.2.1. *The representation* $\boldsymbol{\partial}_{(5,4)}(\boldsymbol{V}) = \boldsymbol{W} = (W_0\,; W_j)_{j\in\partial_{(5,4)}\mathcal{S}}$ *is given as follows, where every quotient space* $\overline{W}$ *stands for* $W/\langle e_3\rangle$.

$\boldsymbol{\partial}_{(5,4)}(\boldsymbol{V})$ :



## 3.3. A proof of Kleiner's Theorem

It was previously shown that posets of finite representation type may be described via Kleiner's classification theorem [Kle75]. According to the statement, a poset $\mathcal{S}$ is of finite representation type if and only if it contains none of Kleiner's critical posets $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$ or $\mathcal{K}_5$ as a subposet. The following preposition plays a fundamental role in proving it.

**Proposition 3.3.1 ([Arn00], Proposition 1.3.1).** *Let $\mathcal{S}$ be a finite poset and $\mathcal{T}$ a subposet of it.*

***i.*** *$\mathcal{S}$ has finite representation type $\Longrightarrow$ $\mathcal{T}$ has finite representation type.*

***ii.*** *$\mathcal{T}$ has infinite representation type $\Longrightarrow$ $\mathcal{S}$ has infinite representation type.*

*Sketch of Proof.* We define the correspondences $F^+ : \mathfrak{sp}_{\mathcal{T}} \longrightarrow \mathfrak{sp}_{\mathcal{S}}$ and $F^- : \mathfrak{sp}_{\mathcal{T}} \longrightarrow \mathfrak{sp}_{\mathcal{S}}$ as follows. Let $\boldsymbol{W} = (W_0; W_i)_{i\in\mathcal{T}} \in \mathfrak{sp}_{\mathcal{T}}$.

$F^+(\boldsymbol{W}) = \boldsymbol{W}^+ = (W_0^+; W_i^+)_{i\in\mathcal{S}}$ is such that

$$W_0^+ = W_0$$

$$W_i^+ = \begin{cases} W_i \text{ if } i \in \mathcal{T} \\ \sum_{\substack{j\prec i \\ j\in\mathcal{T}}} W_j \text{ if } i \in \mathcal{S}\setminus\mathcal{T} \text{ and there exists } j \in \mathcal{T} \text{ such that } j \prec i \\ 0 \text{ otherwise.} \end{cases}$$

Analogously, $F^-(\boldsymbol{W}) = \boldsymbol{W}^- = (W_0^-; W_i^-)_{i\in\mathcal{S}}$ is such that

$$W_0^- = W_0$$

$$W_i^- = \begin{cases} W_i \text{ if } i \in \mathcal{T} \\ \bigcap_{\substack{j\succ i \\ j\in\mathcal{T}}} W_j \text{ if } i \in \mathcal{S}\setminus\mathcal{T} \text{ and there exists } j \in \mathcal{T} \text{ such that } j \succ i \\ W_0 \text{ otherwise.} \end{cases}$$

Given $f : \boldsymbol{W} \longrightarrow \boldsymbol{V}$ a morphism in $\mathfrak{sp}_{\mathcal{T}}$, we naturally define $F^+(f) = f$ and $F^-(f) = f$ as morphisms in $\mathfrak{sp}_{\mathcal{S}}$. Then $F^+ : \mathrm{Hom}(\boldsymbol{W}, \boldsymbol{V}) \longrightarrow \mathrm{Hom}(F^+(\boldsymbol{W}), F^+(\boldsymbol{V}))$ and $F^- : \mathrm{Hom}(\boldsymbol{W}, \boldsymbol{V}) \longrightarrow \mathrm{Hom}(F^-(\boldsymbol{W}), F^-(\boldsymbol{V}))$ consist of isomorphisms for each pair of objects $\boldsymbol{W}, \boldsymbol{V}$ in $\mathfrak{sp}_{\mathcal{T}}$, thus inducing an injection $\mathrm{Ind}(\mathfrak{sp}_{\mathcal{T}}) \longmapsto \mathrm{Ind}(\mathfrak{sp}_{\mathcal{S}})$. □

One last crucial statement is necessary for our task here. Although similar to Nazarova and Roiter's case (*Theorem 2.1.3*), Zavadskij's result below is somehow more precise.

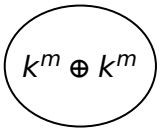
**Theorem 3.3.2 ([Sim92] Theorem 9.10).** *Let $\mathcal{S}$ be a $(a,b)$–differentiable poset. Then the differentiation maps $\boldsymbol{\partial}_{(a,b)}$ yields*

$$\#\operatorname{Ind}\mathfrak{sp}_{\mathcal{S}} = 1 + |\mathcal{S}_b^a| + \#\operatorname{Ind}\mathfrak{sp}_{\partial_{(a,b)}\mathcal{S}}$$

*in case both $\#\operatorname{Ind}\mathfrak{sp}_{\mathcal{S}}$ and $\#\operatorname{Ind}\mathfrak{sp}_{\partial_{(a,b)}\mathcal{S}}$ are finite.*

This theorem's proof relies on a *section map* $\boldsymbol{\partial}^{-}_{(a,b)} : \mathfrak{sp}_{\partial_{(a,b)}\mathcal{S}} \dashrightarrow \mathfrak{sp}_{\mathcal{S}}$ with the property that $\boldsymbol{\partial}_{(a,b)}\boldsymbol{\partial}^{-}_{(a,b)}(\boldsymbol{W}) \cong \boldsymbol{W}$ for every indecomposable representation $\boldsymbol{W} \in \mathfrak{sp}_{\partial_{(a,b)}\mathcal{S}}$. Further details can be found in Simson's textbook ([Sim92], Sections 9.2, 9.3 and 9.4).

We now get down to business by first proving that $K_1 = (1,1,1,1)$ is of infinite representation type ([Arn00], Example 1.1.6). First, let's consider a matrix $A \in \mathbb{M}_{m\times m}(k)$ for $m \in \mathbb{Z}^+$. We should also define $(1+A)\,k^m = \{(x, Ax) \mid x \in k^m\}$, thus obtaining a representation $\boldsymbol{V}_A \in \mathfrak{sp}_{K_1}$:




$$\boldsymbol{V}_A: \qquad k^m \oplus 0 \qquad\qquad 0 \oplus k^m \qquad\qquad (1+1)\,k^m \qquad\qquad (1+A)\,k^m$$

We want to show that there exist infinitely many indecomposable representations of the form $\boldsymbol{V}_A$. Given $\lambda \in k$ and $m \leq 1$, one may consider $A(m,\lambda)$, the $m \times m$ Jordan block matrix with minimal polynomial $(x-\lambda)^m$:

$$A(m,\lambda) = \begin{bmatrix} \lambda & 1 & & \mathbf{0} \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & \lambda \end{bmatrix}.$$

We then observe that $\boldsymbol{V}_{A(m,\lambda)}$ is an indecomposable $k[x]$–module such that $A(m,\lambda) \sim A(m',\lambda') \iff m = m'$ and $\lambda = \lambda'$, thus concluding that

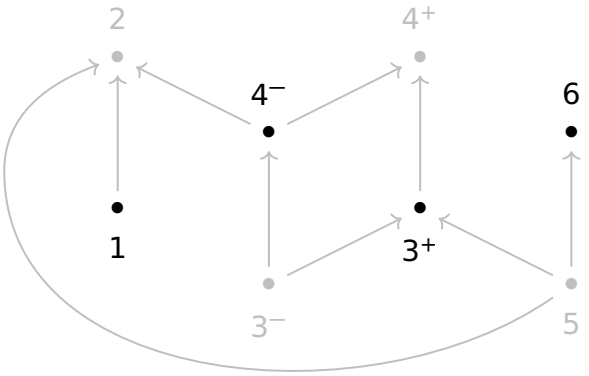
$$\{\boldsymbol{V}_A \mid A = A(m,\lambda), m \geq 1, \lambda \in k\}$$

is an infinite subset of Ind $\mathfrak{sp}_{K_1}$.

The critical posets $K_2, K_3, K_4$ and $K_5$ will be proven of infinite representation type by applying the suitable–pair differentiation algorithm and observing *Proposition 3.3.1*.

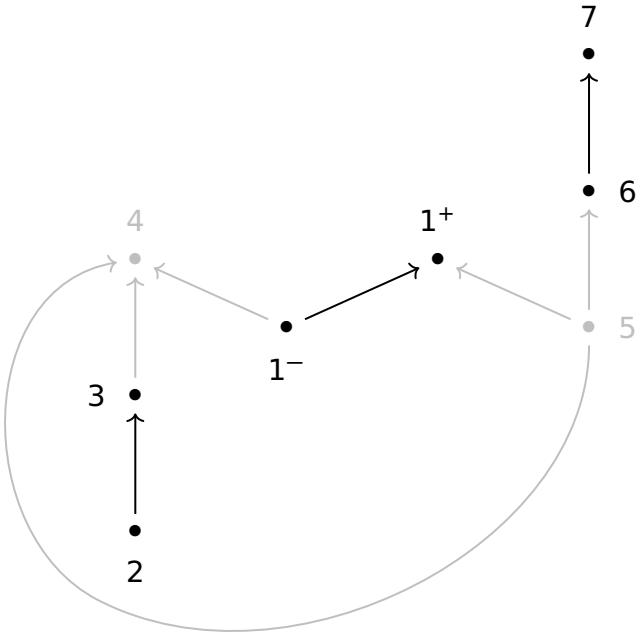
$\partial_{(5,2)}K_2$ contains $\{1, 4^-, 3^+, 6\} = K_1$ as a subposet:

$\partial_{(5,2)}K_2$ :



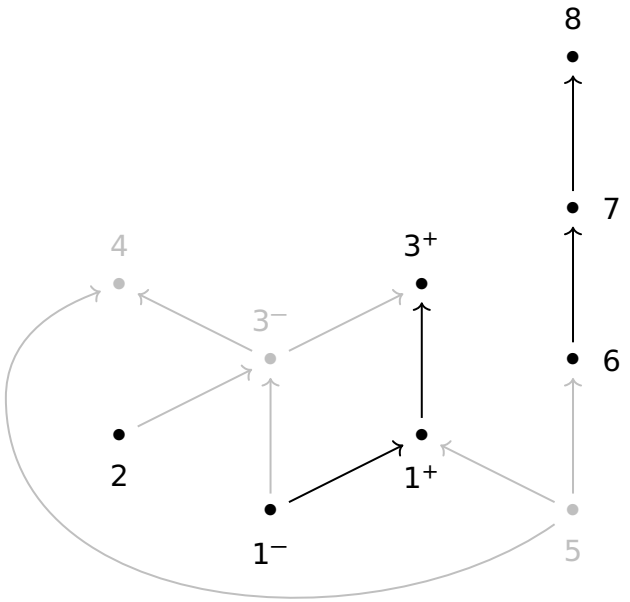

$\partial_{(5,4)}K_3$ contains $\{2 \prec 3, 1^- \prec 1^+, 6 \prec 7\} = K_2$ as a subposet:

$\partial_{(5,4)}K_3$ :



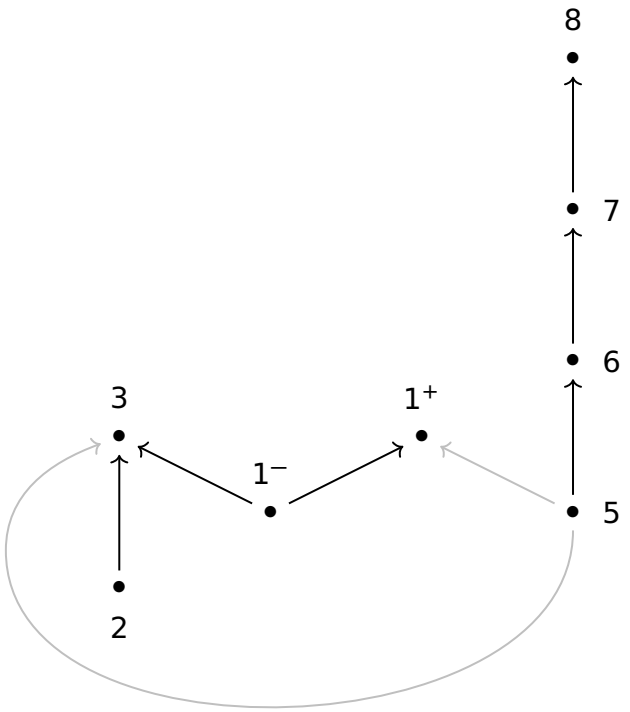
$\partial_{(5,4)}K_4$ contains $\{2, 1^- \prec 1^+ \prec 3^+, 6 \prec 7 \prec 8\} = K_3$ as a subposet:

$\partial_{(5,4)}K_4$ :




$\partial_{(4,3)}K_5$ contains $\{2 \prec 3 \succ 1^- \prec 1^+, 5 \prec 6 \prec 7 \prec 8\} = K_4$ as a subposet:

$\partial_{(4,3)}K_5$ :




We then conclude that if $\mathcal{S}$ contains any of Kleiner's critical posets, then it is of infinite representation type. Proving the converse takes a rather lengthy path, as it involves combinatorial lemmas which guarantee that, if $\mathcal{S}$ doesn't contain any of the critial posets, then there exists a suitable pair $(a, b)$ in $\mathcal{S}$ such that $|\mathcal{S}_b^a| \leq 1$ and $\partial_{(a,b)}\mathcal{S}$ does not contain them either ([Sim92], Sections 10.2 and 10.3).

# Chapter 4

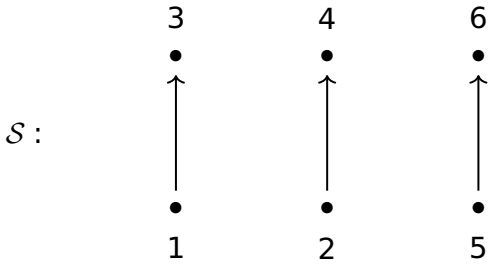
# Some Results

The path to observing whether stability is preserved or not after a suitable–pair differentiation runs through coming up with a matrix that tells us how a poset's dimension vector is transformed by the associated map. Interestingly, this very same matrix has shown inverse Cartan matrices congruence under certain conditions. Although a transformation that precisely converts a stable poset representation's weight into its suitable–pair differentiated one hasn't been achieved yet, some useful lemmas help us build it case-by-case.

## 4.1. On the Congruence of $C_{\mathcal{S}}^{-1}$ and $C_{\partial_{(a,b)}\mathcal{S}}^{-1}$

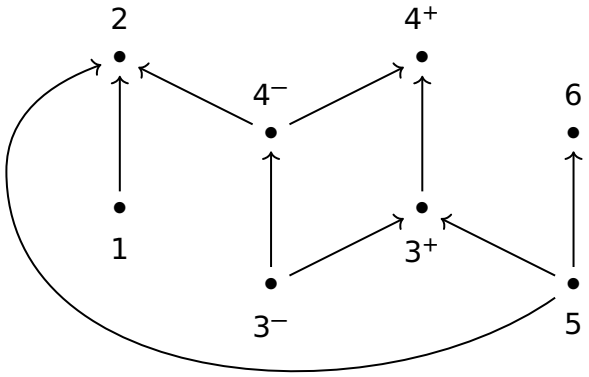
Fonseca's work showed us that describing the way a stable poset representation's weight changes under a maximal–element differentiation is possible by means of a linear transformation ([Fon23], Theorem 4.2.4). Here, a similar approach is attempted for the suitable–pair differentiation procedure.

**Example 4.1.1.** *Consider the poset* $\mathcal{S} = (2, 2, 2)$*.*




$\mathcal{S}$ *is* $(5, 2)$*–differentiable and its differential poset with respect to this suitable pair is given by:*

$\partial_{(5,2)}\mathcal{S}$ :




*The Cartan matrices of $\mathcal{S}*$ and $\partial_{(5,2)}\mathcal{S}*$ are, respectively,*

$C_{\mathcal{S}*} =$

| | $\{0\}$ | $2^{\triangledown}$ | | $S_b^a$ | | $5^{\triangle}$ | |
|---|---|---|---|---|---|---|---|
| $\{0\}$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $2^{\triangledown}$ | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| $S_b^a$ | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| $5^{\triangle}$ | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| | 1 | 0 | 0 | 0 | 0 | 0 | 1 |

$C_{\partial_{(5,2)}\mathcal{S}*} =$

| | $\{0\}$ | $2^{\triangledown}\setminus\{2\}$ | $\{2\}$ | $C^-$ | | $C^+$ | | $\{5\}$ | $5^{\triangle}\setminus\{5\}$ |
|---|---|---|---|---|---|---|---|---|---|
| $\{0\}$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $2^{\triangledown}\setminus\{2\}$ | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\{2\}$ | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C^-$ | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| $C^+$ | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $\{5\}$ | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| $5^{\triangle}\setminus\{5\}$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

*These matrices are invertible and their inverse matrices are*

$$C_{\mathcal{S}*}^{-1} = \left[\begin{array}{c|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

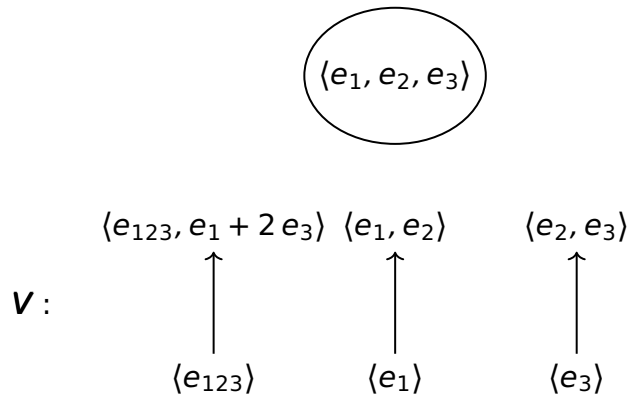
$$C_{\partial_{(5,2)}\mathcal{S}*}^{-1} = \left[\begin{array}{c|c|c|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 2 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & -1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

*Analogously to Fonseca's result* (Example 2.3.1) *there may be constructed a matrix* $T_{\mathcal{S}} \in \mathbb{M}_{7\times 9}(\mathbb{Z})$ *such that* $C_{\mathcal{S}*}^{-1} = T_{\mathcal{S}} \cdot C_{\partial_{(5,2)}\mathcal{S}*}^{-1} \cdot T_{\mathcal{S}}^{tr}$, *i.e. the inverse Cartan matrices are congruent. For this example,* $T_{\mathcal{S}}$ *is given by*

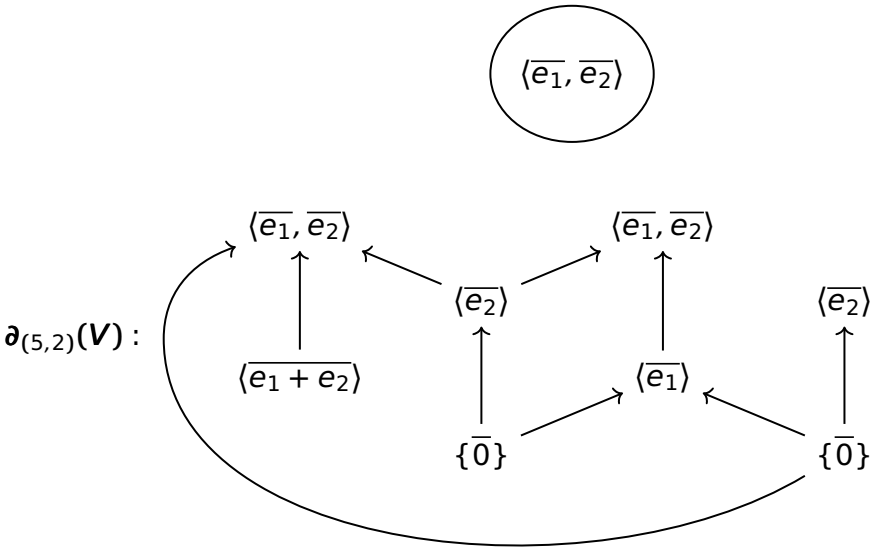
$$T_{\mathcal{S}} = \begin{array}{c|c|c|c|cc|cc|c|c|}
 & \{0\} & 2^{\triangledown}\setminus\{2\} & \{2\} & \multicolumn{2}{c|}{C^{-}} & \multicolumn{2}{c|}{C^{+}} & \{5\} & 5^{\triangle}\setminus\{5\} \\ \hline
\{0\} & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ \hline
2^{\triangledown}\setminus\{2\} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
\{2\} & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline
C & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline
\{5\} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline
5^{\triangle}\setminus\{5\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
\end{array}$$

The particular choice of portraying these matrices as blocks, apart from making it easier to generalise the result, comes from another line of thought. This very same matrix $T_{\mathcal{S}} \in \mathbb{M}_{7\times 9}(\mathbb{Z})$ was initially conceived as a way of changing dimension vectors. Here's an example to clear things up.

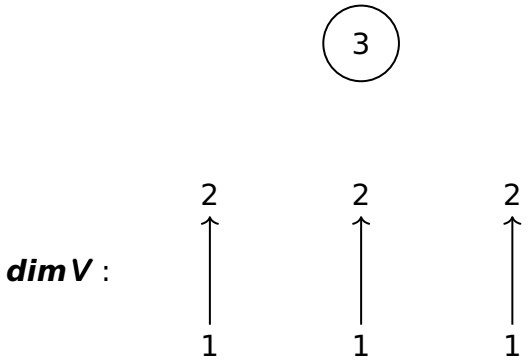
**Example 4.1.2.** *Still with the poset* $\mathcal{S} = (2, 2, 2)$ *from* Example 4.1.1, *consider the following subspace representation* $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$

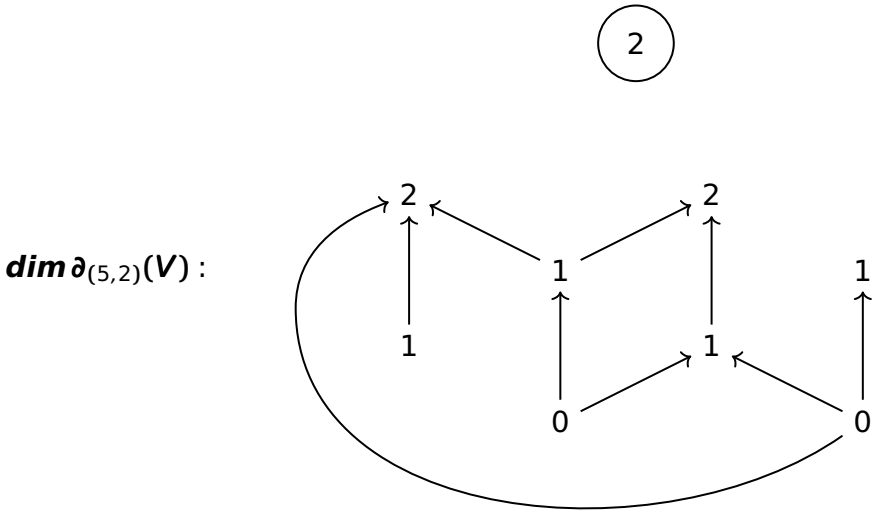



*For the sake of simplicity, it was written* $\langle e_{123}\rangle$ *instead of* $\langle e_1 + e_2 + e_3\rangle$. *We saw that* $\mathcal{S}$ *is* $(5, 2)$*–differentiable and its differential poset* $\partial_{(5,2)}\mathcal{S}$ *was also shown in* Example 4.1.1. *Applying our differentiation map from* Definition 3.2.1, *where the k–subspace* $U \subseteq V_5$ *was chosen to be* $V_5 = \langle e_3\rangle$ *itself, we get that*




*Here, it was written $\overline{W}$ for every isoclass $W/\langle e_3 \rangle$ in order to improve readability. We shall now proceed to analyzing the dimension vectors of both $\boldsymbol{V}$ and $\boldsymbol{\partial}_{(5,2)}(\boldsymbol{V})$.*

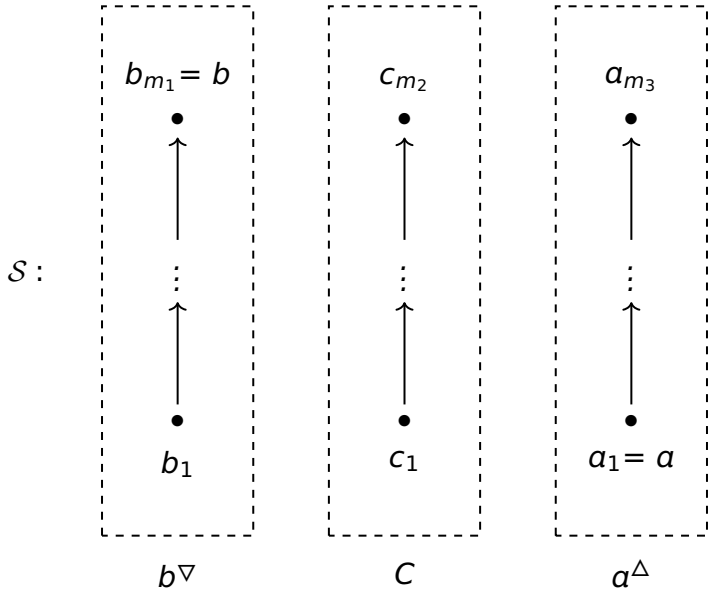



*A straightforward calculation shows that $\boldsymbol{dim}\,\boldsymbol{\partial}_{(5,2)}(\boldsymbol{V}) = \boldsymbol{dim}\,\boldsymbol{V} \cdot T_{\mathcal{S}}$.*

Now that the motivation behind the conception of the matrix $T_{\mathcal{S}}$ has been given, we may go on to a generalisation of the foregoing ideas.

**Proposition 4.1.3.** *Let $\mathcal{S}$ be a finite poset consisting of three chains.*




*Then $\mathcal{S}$ is $(a,b)$–differentiable and $C_{\mathcal{S}*}^{-1} = T_{\mathcal{S}} \cdot C_{\partial_{(a,b)}\mathcal{S}*}^{-1} \cdot T_{\mathcal{S}}^{tr}$, where the matrix $T_{\mathcal{S}} \in \mathbb{M}_{|\mathcal{S}*|\times|\partial_{(a,b)}\mathcal{S}*|}(\mathbb{Z})$ is given by*

$T_{\mathcal{S}} =$

| | $\{0\}$ | $b^{\nabla}\setminus\{b\}$ | $\{b\}$ | $C^{-}$ | $C^{+}$ | $\{a\}$ | $a^{\Delta}\setminus\{a\}$ |
|---|---|---|---|---|---|---|---|
| $\{0\}$ | $1$ | $\mathbf{0}_{1\times(m_1-1)}$ | $0$ | $-E_{m_2}^{tr}$ | $\mathbf{0}_{1\times m_2}$ | $0$ | $\mathbf{0}_{1\times(m_3-1)}$ |
| $b^{\nabla}\setminus\{b\}$ | $\mathbf{0}_{(m_1-1)\times 1}$ | $\mathrm{Id}_{m_1-1}$ | $\mathbf{0}_{(m_1-1)\times 1}$ | $\mathbf{0}_{(m_1-1)\times m_2}$ | $\mathbf{0}_{(m_1-1)\times m_2}$ | $\mathbf{0}_{(m_1-1)\times 1}$ | $\mathbf{0}_{(m_1-1)\times(m_3-1)}$ |
| $\{b\}$ | $0$ | $\mathbf{0}_{1\times(m_1-1)}$ | $1$ | $E_{m_2}^{tr}$ | $\mathbf{0}_{1\times m_2}$ | $0$ | $\mathbf{0}_{1\times(m_3-1)}$ |
| $C$ | $\mathbf{0}_{m_2\times 1}$ | $\mathbf{0}_{m_2\times(m_1-1)}$ | $\mathbf{0}_{m_2\times 1}$ | $\mathrm{Id}_{m_2}$ | $\mathrm{Id}_{m_2}$ | $\mathbf{0}_{m_2\times 1}$ | $\mathbf{0}_{m_2\times(m_3-1)}$ |
| $\{a\}$ | $-1$ | $\mathbf{0}_{1\times(m_1-1)}$ | $0$ | $\mathbf{0}_{1\times m_2}$ | $\mathbf{0}_{1\times m_2}$ | $0$ | $-E_{m_3-1}^{tr}$ |
| $a^{\Delta}\setminus\{a\}$ | $\mathbf{0}_{(m_3-1)\times 1}$ | $\mathbf{0}_{(m_3-1)\times(m_1-1)}$ | $\mathbf{0}_{(m_3-1)\times 1}$ | $\mathbf{0}_{(m_3-1)\times m_2}$ | $\mathbf{0}_{(m_3-1)\times m_2}$ | $\mathbf{0}_{(m_3-1)\times 1}$ | $\mathrm{Id}_{m_3-1}$ |

*Proof.* The first step consists of writing the extended poset's Cartan matrix as a block matrix.

$$C_{\mathcal{S}*} = \begin{array}{c} \\ \{0\} \\ b^{\triangledown} \\ C \\ a^{\triangle} \end{array} \begin{array}{c} \begin{array}{cccc} \{0\} & b^{\triangledown} & C & a^{\triangle} \end{array} \\ \left[\begin{array}{c|c|c|c} 1 & \mathbf{0}_{1\times m_1} & \mathbf{0}_{1\times m_2} & \mathbf{0}_{1\times m_3} \\ \hline E_{m_1} & X_{m_1} & \mathbf{0}_{m_1\times m_2} & \mathbf{0}_{m_1\times m_3} \\ \hline E_{m_2} & \mathbf{0}_{m_2\times m_1} & X_{m_2} & \mathbf{0}_{m_2\times m_3} \\ \hline E_{m_3} & \mathbf{0}_{m_3\times m_1} & \mathbf{0}_{m_3\times m_2} & X_{m_3} \end{array}\right] \end{array}$$

Given $n \in \mathbb{N}$, the matrices $E_n \in \mathbb{M}_{n\times 1}(\mathbb{Z})$ and $X_n \in \mathbb{M}_{n\times n}(\mathbb{Z})$ stand for

$$E_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \text{and} \qquad X_n = \begin{bmatrix} 1 & 1 & \dots & 1 \\ & 1 & \dots & 1 \\ & & \ddots & \vdots \\ \mathbf{0} & & & 1 \end{bmatrix}.$$

$C_{\mathcal{S}*}$ is an upper block triangular matrix where all of its diagonal entries are nonzero. Not only does its inverse matrix exists, but also it may be easily calculated:

$$C_{\mathcal{S}*}^{-1} = \left[\begin{array}{c|c|c|c} 1 & \mathbf{0}_{1\times m_1} & \mathbf{0}_{1\times m_2} & \mathbf{0}_{1\times m_3} \\ \hline -F_{m_1} & X_{m_1}^{-1} & \mathbf{0}_{m_1\times m_2} & \mathbf{0}_{m_1\times m_3} \\ \hline -F_{m_2} & \mathbf{0}_{m_2\times m_1} & X_{m_2}^{-1} & \mathbf{0}_{m_2\times m_3} \\ \hline -F_{m_3} & \mathbf{0}_{m_3\times m_1} & \mathbf{0}_{m_3\times m_2} & X_{m_3}^{-1} \end{array}\right].$$

For any $n \in \mathbb{N}$, the matrices $F_n \in \mathbb{M}_{n\times 1}(\mathbb{Z})$ and $X_n^{-1} \in \mathbb{M}_{n\times n}(\mathbb{Z})$ respectively denote

$$F_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad X_n^{-1} = \begin{bmatrix} 1 & -1 & & & \mathbf{0} \\ & 1 & -1 & & \\ & & 1 & \ddots & \\ & & & \ddots & -1 \\ \mathbf{0} & & & & 1 \end{bmatrix}.$$

By doing the very same procedure, we obtain the extended differential poset's Cartan matrix and its inverse one written as block matrices.

$C_{\partial_{(a,b)}\mathcal{S}*} =$

| | $\{0\}$ | $b^{\triangledown} \setminus \{b\}$ | $\{b\}$ | $C^{-}$ | $C^{+}$ | $\{a\}$ | $a^{\triangle} \setminus \{a\}$ |
|---|---|---|---|---|---|---|---|
| $\{0\}$ | $1$ | $\mathbf{0}_{1\times(m_1-1)}$ | $0$ | $\mathbf{0}_{1\times m_2}$ | $\mathbf{0}_{1\times m_2}$ | $0$ | $\mathbf{0}_{1\times(m_3-1)}$ |
| $b^{\triangledown} \setminus \{b\}$ | $E_{m_1-1}$ | $X_{m_1-1}$ | $E_{m_1-1}$ | $\mathbf{0}_{(m_1-1)\times m_2}$ | $\mathbf{0}_{(m_1-1)\times m_2}$ | $\mathbf{0}_{(m_1-1)\times 1}$ | $\mathbf{0}_{(m_1-1)\times(m_3-1)}$ |
| $\{b\}$ | $1$ | $\mathbf{0}_{1\times(m_1-1)}$ | $1$ | $\mathbf{0}_{1\times m_2}$ | $\mathbf{0}_{1\times m_2}$ | $0$ | $\mathbf{0}_{1\times(m_3-1)}$ |
| $C^{-}$ | $E_{m_2}$ | $\mathbf{0}_{m_2\times(m_1-1)}$ | $E_{m_2}$ | $X_{m_2}$ | $X_{m_2}$ | $\mathbf{0}_{m_2\times 1}$ | $\mathbf{0}_{m_2\times(m_3-1)}$ |
| $C^{+}$ | $E_{m_2}$ | $\mathbf{0}_{m_2\times(m_1-1)}$ | $\mathbf{0}_{m_2\times 1}$ | $\mathbf{0}_{m_2\times m_2}$ | $X_{m_2}$ | $\mathbf{0}_{m_2\times 1}$ | $\mathbf{0}_{m_2\times(m_3-1)}$ |
| $\{a\}$ | $1$ | $\mathbf{0}_{1\times(m_1-1)}$ | $1$ | $\mathbf{0}_{1\times m_2}$ | $E_{m_2}^{tr}$ | $1$ | $E_{m_3-1}^{tr}$ |
| $a^{\triangle} \setminus \{a\}$ | $E_{m_3-1}$ | $\mathbf{0}_{(m_3-1)\times(m_1-1)}$ | $\mathbf{0}_{(m_3-1)\times 1}$ | $\mathbf{0}_{(m_3-1)\times m_2}$ | $\mathbf{0}_{(m_3-1)\times m_2}$ | $\mathbf{0}_{(m_3-1)\times 1}$ | $X_{m_3-1}$ |

$C_{\partial_{(a,b)}\mathcal{S}*}^{-1} =$

| | | | | | | |
|---|---|---|---|---|---|---|
| $1$ | $\mathbf{0}_{1\times(m_1-1)}$ | $0$ | $\mathbf{0}_{1\times m_2}$ | $\mathbf{0}_{1\times m_2}$ | $0$ | $\mathbf{0}_{1\times(m_3-1)}$ |
| $\mathbf{0}_{(m_1-1)\times 1}$ | $X_{m_1-1}^{-1}$ | $-F_{m_1-1}$ | $\mathbf{0}_{(m_1-1)\times m_2}$ | $\mathbf{0}_{(m_1-1)\times m_2}$ | $\mathbf{0}_{(m_1-1)\times 1}$ | $\mathbf{0}_{(m_1-1)\times(m_3-1)}$ |
| $-1$ | $\mathbf{0}_{1\times(m_1-1)}$ | $1$ | $\mathbf{0}_{1\times m_2}$ | $\mathbf{0}_{1\times m_2}$ | $0$ | $\mathbf{0}_{1\times(m_3-1)}$ |
| $F_{m_2}$ | $\mathbf{0}_{m_2\times(m_1-1)}$ | $-F_{m_2}$ | $X_{m_2}^{-1}$ | $X_{m_2}^{-1}$ | $\mathbf{0}_{m_2\times 1}$ | $\mathbf{0}_{m_2\times(m_3-1)}$ |
| $-F_{m_2}$ | $\mathbf{0}_{m_2\times(m_1-1)}$ | $\mathbf{0}_{m_2\times 1}$ | $\mathbf{0}_{m_2\times m_2}$ | $X_{m_2}^{-1}$ | $\mathbf{0}_{m_2\times 1}$ | $\mathbf{0}_{m_2\times(m_3-1)}$ |
| $2$ | $\mathbf{0}_{1\times(m_1-1)}$ | $-1$ | $\mathbf{0}_{1\times m_2}$ | $-F_{m_2}^{tr}$ | $1$ | $-F_{m_3-1}^{tr}$ |
| $-F_{m_3-1}$ | $\mathbf{0}_{(m_3-1)\times(m_1-1)}$ | $\mathbf{0}_{(m_3-1)\times 1}$ | $\mathbf{0}_{(m_3-1)\times m_2}$ | $\mathbf{0}_{(m_3-1)\times m_2}$ | $\mathbf{0}_{(m_3-1)\times 1}$ | $X_{m_3-1}^{-1}$ |

By multiplying the block matrices, we show that the equality holds:

$$C^{-1}_{\mathcal{S}*} = T_{\mathcal{S}} \cdot C^{-1}_{\partial_{(a,b)}\mathcal{S}*} \cdot T^{tr}_{\mathcal{S}}.$$

□

## 4.2. Stability after a suitable–pair differentiation

Although a linear transformation performing the weight change after a suitable–pair differentiation still hasn't been achieved, a lemma can help us. It had been observed that the suitable–pair differentiation map yields a subspace representation which usually contains the ambient space itself or a copy of another subspace. That's why the result below might be useful in numerous cases.

**Lemma 4.2.1.** *Let $\mathcal{S}$ be a finite poset with elements $\{1,\dots,n\}$ and $\boldsymbol{V} \in \mathfrak{sp}_{\mathcal{S}}$ be a $\boldsymbol{\theta}$–stable representation for a certain $\boldsymbol{\theta} \in \mathbb{Z}^{\mathcal{S}*}$. Consider $\mathcal{T} = \mathcal{S} \cup \{n+1\}$, the extension of $\mathcal{S}$ by the element $n+1$.*

***i.*** *If $\boldsymbol{W} \in \mathfrak{sp}_{\mathcal{T}}$ is such that*

$$\begin{aligned} W_0 &= V_0, \\ W_1 &= V_1, \\ &\vdots \\ W_n &= V_n \text{ and} \\ W_{n+1} &= V_0, \end{aligned}$$

*then $\boldsymbol{W}$ is $\boldsymbol{\alpha}$–stable for*

$$\boldsymbol{\alpha} = \begin{bmatrix} \theta_0 - m & \theta_1 & \cdots & \theta_n & m \end{bmatrix} \in \mathbb{Z}^{\mathcal{T}*}$$

*and every $m \in \mathbb{Z}^+$.*

***ii.*** *If* $\boldsymbol{W} \in \mathfrak{sp}_{\mathcal{T}}$ *is such that*

$$\begin{aligned} W_0 &= V_0, \\ W_1 &= V_1, \\ &\vdots \\ W_n &= V_n \text{ and} \\ W_{n+1} &= V_n, \end{aligned}$$

*then* $\boldsymbol{W}$ *is* $\boldsymbol{\beta}$*–stable, where*

$$\boldsymbol{\beta} = \begin{bmatrix} 2\theta_0 & 2\theta_1 & \cdots & 2\theta_{n-1} & \theta_n & \theta_n \end{bmatrix} \in \mathbb{Z}^{\mathcal{T}}{}^{*}.$$

*Proof.*

**i.** We first check that

$$\begin{aligned} \boldsymbol{\alpha} \cdot (\mathbf{dim}\, \boldsymbol{W})^{tr} &= \begin{bmatrix} \theta_0 - m & \theta_1 & \cdots & \theta_n & m \end{bmatrix} \cdot \begin{bmatrix} \dim V_0 \\ \dim V_1 \\ \vdots \\ \dim V_n \\ \dim V_0 \end{bmatrix} \\ &= (\theta_0 - m) \dim V_0 + \sum_{i=1}^{n} \theta_i \dim V_i + m \dim V_0 \\ &= \sum_{j=0}^{n} \theta_j \dim V_j - m \dim V_0 + m \dim V_0 \\ &= 0. \end{aligned}$$

Any proper subrepresentation of $\boldsymbol{W}$ has the form

$$\boldsymbol{W}_K = (V_0 \cap K; V_1 \cap K, \ldots V_n \cap K, V_0 \cap K)$$

for $K$ a proper subspace of $V_0$. We may note that $\boldsymbol{V}_K = (V_0 \cap K; V_i \cap K)_{i=1}^{n}$ is also a proper subrepresentation of $\boldsymbol{V}$, which allows us to conclude that:

$$\boldsymbol{\alpha}\cdot(\mathbf{dim}\,\boldsymbol{W}_K)^{tr}=\begin{bmatrix}\theta_0-m & \theta_1 & \cdots & \theta_n & m\end{bmatrix}\cdot\begin{bmatrix}\dim(V_0\cap K)\\ \dim(V_1\cap K)\\ \vdots\\ \dim(V_n\cap K)\\ \dim(V_0\cap K)\end{bmatrix}$$

$$=(\theta_0-m)\dim(V_0\cap K)+\sum_{i=1}^{n}\theta_i\dim(V_i\cap K)+m\dim(V_0\cap K)$$

$$=\sum_{j=0}^{n}\theta_j\dim(V_j\cap K)-m\dim(V_0\cap K)+m\dim(V_0\cap K)$$

$$<0.$$

**ii.** We begin our proof by checking that

$$\boldsymbol{\beta}\cdot(\mathbf{dim}\,\boldsymbol{W})^{tr}=\begin{bmatrix}2\theta_0 & \cdots & 2\theta_{n-1} & \theta_n & \theta_n\end{bmatrix}\cdot\begin{bmatrix}\dim V_0\\ \vdots\\ \dim V_{n-1}\\ \dim V_n\\ \dim V_n\end{bmatrix}$$

$$=\sum_{i=0}^{n-1}2\,\theta_i\dim V_i+2\,\theta_n\dim V_n=2\sum_{j=0}^{n}\theta_j\dim V_j=0.$$

Once again, we consider any proper subrepresentation of $\boldsymbol{W}$, which must have the form

$$\boldsymbol{W}_K=(V_0\cap K;V_1\cap K,\ldots V_n\cap K,V_n\cap K)$$

for a proper subspace $K$ of $V_0$. If we spot that $\boldsymbol{V}_K=(V_0\cap K;V_i\cap K)_{i=1}^{n}$ consists of a proper subrepresentation of $\boldsymbol{V}$, we may then wrap it up by checking that

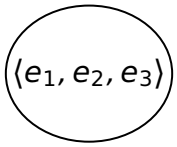
$$\boldsymbol{\beta}\cdot(\mathbf{dim}\,\boldsymbol{W}_K)^{tr}=\begin{bmatrix} 2\,\theta_0 & \cdots & 2\,\theta_{n-1} & \theta_n & \theta_n \end{bmatrix}\cdot\begin{bmatrix} \dim(V_0\cap K)\\ \vdots \\ \dim(V_{n-1}\cap K)\\ \dim(V_n\cap K)\\ \dim(V_n\cap K)\end{bmatrix}$$

$$=\sum_{i=0}^{n-1}2\,\theta_i\dim(V_i\cap K)+2\,\theta_n\dim(V_n\cap K)$$

$$=2\sum_{j=0}^{n}\theta_j\dim(V_j\cap K)<0.$$

□

**Example 4.2.2.** *Let $\mathcal{S}=(1,1,1)$ and consider the following representation $\boldsymbol{V}\in\mathfrak{sp}_{\mathcal{S}}$:*



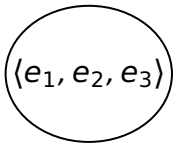

$\boldsymbol{V}:\qquad \langle e_{123}\rangle \qquad \langle e_1\rangle \qquad \langle e_3\rangle$

*Then $\boldsymbol{V}$ is $\boldsymbol{\theta}$–stable, where $\boldsymbol{\theta}\in\mathbb{Z}^{\mathcal{S}}{}^{*}$ is given by*




$\boldsymbol{\theta}:\qquad 3 \qquad 3 \qquad 3$

*Let $\mathcal{T}=(1,1,1,1)$, an extension of $\mathcal{S}$, and $\boldsymbol{W}_1\in\mathfrak{sp}_{\mathcal{T}}$ such that*



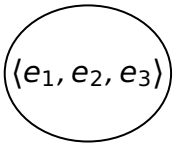

$\boldsymbol{W}_1$ : $\langle e_{123} \rangle$ $\langle e_1 \rangle$ $\langle e_3 \rangle$ $\langle e_1, e_2, e_3 \rangle$

*Applying* Lemma 4.2.1, *we get that* $\boldsymbol{W}_1$ *is* $\boldsymbol{\alpha}$*–stable for* $\boldsymbol{\alpha} \in \mathbb{Z}^{\mathcal{T}}{}^*$ *given by*




$\boldsymbol{\alpha}$ : 3 3 3 1

*Similarly, if* $\boldsymbol{W}_2 \in \mathfrak{sp}_{\mathcal{T}}$ *is*




$\boldsymbol{W}_2$ : $\langle e_{123} \rangle$ $\langle e_1 \rangle$ $\langle e_3 \rangle$ $\langle e_3 \rangle$

*then* $\boldsymbol{W}_2$ *is* $\boldsymbol{\beta}$*–stable, where* $\boldsymbol{\beta} \in \mathbb{Z}^{\mathcal{T}}{}^*$ *is*




$\boldsymbol{\beta}$ : 6 6 3 3

# Appendix A

Zavadskij's article in which he first defined a suitable–pair differentiation algorithm was published in 1977 [Zav77]. Not only is the text is Russian, but also the book containing it is somewhat rar and unavailable on the internet. For this reason, with a slight knowledge of *Cyrillic script*, the assistance of translating apps and professor Iusenko's revision, a *free translation* of it was chosen to be added below. Original terms and notations were kept, although equations were depicted in a more intelligible way.

### Differentiation by a Pair of Points (A. G. Zavadskij)

Recall (see [1], [2], [3]) that a representation of a finite partially ordered set $\mathcal{M}$ over a field $k$ is a set of finite-dimensional $k$–spaces of the form $\{V; V_x \mid x \in \mathcal{M}\}$ where $V \supset V_x$ for any $x \in \mathcal{M}$ and $V_x \subset V_y$ for $x \preceq y$. Indecomposable and equivalent representations of a set are defined in a natural way.

In [1], an algorithm for differentiating partially ordered sets and their representations was constructed, which was then used to prove the criterion for the finiteness of a partially ordered set type [4], to solve the Brauer-Thrall problem for algebras over a perfect field [5], to study the representations of quivers [3], to describe quivers and partially ordered sets of tame type [2], [6], and to consider other issues. The condition that allows this algorithm to be applied is the existence of a maximal or minimal point in a partially ordered set that is not included in the four incomparable elements.

This article describes a new algorithm for differentiating partially ordered sets, which is applicable in a more general situation.

We first describe the formal procedure of differentiation. We assume that all partially ordered sets are finite.

We will say that a partially ordered set $\mathcal{M}$ is the sum of sets $A_1, \ldots, A_n$ and write $\mathcal{M} = A_1 + \cdots + A_n$ if the sets $A_1, \ldots, A_n$ are pairwise disjoint subsets of the set $\mathcal{M}$, the union of which coincides with $\mathcal{M}$. Of course, elements of different sets $A_1, \ldots, A_n$ can be comparable with each other.

We denote by $\{x\}^{\Delta}$ and $\{x\}^{\nabla}$, respectively, the upper and lower cones of the element $x$: $\{x\}^{\Delta} = \{y \in \mathcal{M} \mid y \succeq x\}$, $\{x\}^{\nabla} = \{y \in \mathcal{M} \mid y \preceq x\}$.

A pair of points $(a, b)$, $a \neq b$, of an arbitrary partially ordered set $\mathcal{M}$ will be called suitable if the following two conditions are satisfied:

1) $a \not\succ b$;

2) $\mathcal{M} = \{a\}^{\Delta} + \{b\}^{\nabla} + C$, where $C$ is a linearly ordered set (possibly empty).

The derived set of a set with respect to a suitable pair of points $(a, b)$ is the set $\mathcal{M}'_{(a,b)}$, obtained from $\mathcal{M}$ as follows:

1) the linearly ordered set $C = \{c_1 \prec c_2 \prec \cdots \prec c_n\}$ is replaced by two new linearly ordered sets $C^- = \{c_1^- \prec c_2^- \prec \cdots \prec c_n^-\}$ and $C^+ = \{c_1^+ \prec c_2^+ \prec \cdots \prec c_n^+\}$. Moreover, for any $1 \preceq i \preceq n$, $c_i^- \prec c_i^+$ and each of the points $c_i^-, c_i^+$ inherits all the connections that the point $c_i$ had with the points of the set $\mathcal{M} \setminus C$;

2) the connections $a \prec b$, $a \prec c_1^+$, $c_n^- \prec b$ and, of course, the connections induced by these connections;

3) if points $x, y$ arise such that $x \prec y$ and $y \prec x$ , then $x$ and $y$ are identified. It follows from the definition that in the case of $C = \varnothing$ differentiation consists of making only one additional connection $a \prec b$ (with the possible identification $a = b$ if initially $a \succ b$).

Now let us explain how the algorithm operates on representations.

Let $\boldsymbol{T} = \{V; V_x \mid x \in \mathcal{M}\}$ be a representation of a partially ordered set $\mathcal{M}$ with an appropriate pair of points $(a, b)$. Let $V_a + V_b = U \oplus V_b$ where $U \subset V_a$. We associate with the representation $\boldsymbol{T}$ a representation $\boldsymbol{T}'$ of the derived set $\mathcal{M}'_{(a,b)}$ as follows. Put $\boldsymbol{T}' = \{W; W_x \mid x \in \mathcal{M}'_{(a,b)}\}$ where

$$
\begin{aligned}
W &= V/U;\\
W_x &= (V_x + U)/U \text{ for all } x \in \mathcal{M}'_{(a,b)} \setminus (C^- \cup C^+) \text{ and}\\
W_{c_i^+} &= (V_a + V_{c_i} + U)/U;\\
W_{c_i^-} &= \big((V_b \cap Vc_i) + U\big)/U.
\end{aligned}
$$

We will show that the representation $\boldsymbol{T}'$ defined in this way does not depend, up to isomorphism, on the choice of $U$.

Indeed, let $V = (V_a + V_b) \oplus X = U \oplus V_b \oplus X$ and $\pi : V \longrightarrow V_b \oplus X$ be the natural projection corresponding to this decomposition. Setting

$$\begin{aligned} W &= V_b \oplus X \\ W_x &= \pi(V_x) \text{ for any } x \in \mathcal{M}'_{(a,b)} \setminus (C^+ \cup C^-) \\ W_{c_i^+} &= \pi(V_a + V_{c_i}) \text{ and} \\ W_{c_i^-} &= \pi(V_b \cap V_{c_i}) = V_b \cap V_{c_i}, \end{aligned}$$

we obtain a representation of the derived set $\boldsymbol{T}' = \{W; W_x \mid x \in \mathcal{M}'_{(a,b)}\}$.

Let us now consider another decomposition $V = U_1 \oplus V_b \oplus X$ with a new $U_1$ (but with the same $x$) and the canonical projection $\pi_1 : V \longrightarrow V_b \oplus X$. It corresponds to a similarly defined representation $\boldsymbol{T}'_1 = \{W; W_x^1 \mid x \in \mathcal{M}'_{(a,b)}\}$ of the set $\mathcal{M}'_{(a,b)}$.

Denote $V_a \cap V_b = \Delta$. Since $\Delta \oplus U = \Delta \oplus U_1 = V_a$, then $\pi(v) - \pi_1(v) \in \Delta$ for any $v \in V$. In addition, $\pi(\alpha) = \pi_1(\alpha) = \alpha$ for every $\alpha \in W$. Consequently, $\pi(V_x) = \pi_1(V_x)$ when $x \in \mathcal{M}'_{(a,b)} \setminus (C^+ \cup C^-)$ and $\pi(V_a + V_{c_i}) = \pi_1(V_a + V_{c_i})$, i.e. $W_x = W_x^1$ for every $x \in \mathcal{M}'_{(a,b)}$ and thus $\boldsymbol{T}' = \boldsymbol{T}'_1$. The correctness of the operation of transition from $\boldsymbol{T}'$ to $\boldsymbol{T}'_1$ has been proven.

Now let $\boldsymbol{S} = \{W; W_x \mid x \in \mathcal{M}'_{(a,b)}\}$ be a representation of the derived set $\mathcal{M}'_{(a,b)}$. Let's restore the representation $[\boldsymbol{S}] = \{V; V_x \mid x \in \mathcal{M}\}$ of the set $\mathcal{M}$ for which $[\boldsymbol{S}] \simeq \boldsymbol{S}$. To avoid the need to take identifications into account, we consider the basic case when the points $a$ and $b$ are not comparable with each other and are not comparable with the points of the set $C$.

Let's represent the space $W_{c_i}^+$ as $W_{c_i}^+ = F_i \oplus H_i \oplus \sum_{x \prec c_i^+} W_x$, where $F_i \subset W_b$, $H_i \cap W_b = 0$ and let's designate $f_1^i, \ldots, f_{p_i}^i$ basis of the space $F_i$ $(i \le i \le n)$. Assuming that the symbols $u_j^i$ $(1 \le i \le n)$ form a basis of some vector space $U$, let's put $V = U \oplus W$ and denote $e_j^i = f_j^i + u_j^i$.

Let $V_x = U \oplus W_x$ for any $x \in \{a\}^\triangle$; $V_y = W_y$, for any $y \in \{b\}^\triangledown$. Denoting by $\langle \alpha, \beta, \ldots, \gamma \rangle$ the vector space generated by the vectors $\alpha, \beta, \ldots, \gamma$, and setting $V_{c_1} = \langle e'_1, \ldots, e'_{p_1} \rangle + H_1 + W_{c_1^-}$; $V_{c_i} = \langle e_1^i, \ldots, e_{p_i}^i \rangle + H_i + W_{c_i^-} + V_{c_{i-1}}$ $(i \ge 2)$, we

obtain a representation $[\boldsymbol{S}] = \{V; V_x \mid x \in \mathcal{M}\}$ of the set $\mathcal{M}$ .The construction of the representation $[\boldsymbol{S}]$ in the case of an arbitrary suitable pair of points $(a, b)$ is carried out similarly.

It is easily verified that, up to isomorphism, the representation $[\boldsymbol{S}]$ does not depend on the method of choosing the spaces $F_i, H_i$ and the basis elements $f^i_j$.

For an arbitrary representation $\boldsymbol{T} = \{V; V_x \mid x \in \mathcal{N}\}$ of the set $\mathcal{N}$, we denote by $w(\boldsymbol{T})$ the sum $w(\boldsymbol{T}) = d + \sum_{x\in\mathcal{N}} d_x$, where $d = \dim V$; $d_x = \dim V_x / \sum_{y\prec x} V_y$.

**Proposition 1.** Let be a partially ordered set with a suitable pair of points $(a, b)$. Then

(a) $[\boldsymbol{S}] \simeq \boldsymbol{S}$ for any representation $\boldsymbol{S}$ of the set $\mathcal{M}'_{(a,b)}$;

(b) if $\boldsymbol{T}$ is any representation of the set $\mathcal{M}$ that does not contain indecomposable sets $\{a\}$ and $\{a, c_i\}$ as direct summands, where $c_i \in C$, then $[\boldsymbol{T}'] \simeq \boldsymbol{T}$;

(c) for any representation $\boldsymbol{T}$ of the set $\mathcal{M}$, $w(\boldsymbol{T}') \le w(\boldsymbol{T})$;

(d) for any representation $\boldsymbol{T} = \{V/V_x \mid x \in \mathcal{M}\}$ of the set $\mathcal{M}$ such that $V_a \not\subset V_b$, $w(\boldsymbol{T}') < w(\boldsymbol{T})$.

*Proof.* Items (a), (c) and (d) are fairly obvious, so we prove only item (b).

Let $\boldsymbol{T} = \{V/V_x \mid x \in \mathcal{M}\}$ be a representation of the set $\mathcal{M}$ and $V = U \oplus V_b \oplus X$ where $U \subset V_a$ and $U \oplus V_b = V_a + V_b$. Denote by $\pi_x : V \longrightarrow X$ and $\pi_u : V \longrightarrow U$ the canonical projections corresponding to this decomposition.

Let

$$\begin{aligned} \pi_x(V_{c_1}) &= H_1, \\ \pi_x(V_{c_2}) &= H_1 \oplus H_2, \\ &\vdots \\ \pi_x(V_{c_n}) &= H_1 \oplus \cdots \oplus H_n. \end{aligned}$$

Since the space $X$ is defined modulo $U \oplus V_b$ (as the complement of $U \oplus V_b$ in the decomposition $V = U \oplus V_b \oplus X$), and $H_1, \ldots, H_n$ are mutually complementary

to each other, we can assume that $H_i \subset V_{c_i}$ for any $i$. Thus, if $\alpha \in V_{c_i}$ and $\pi_x(\alpha) \neq 0$, then $\alpha \in X$.

Let

$$\begin{aligned}
\pi_u(V_{c_1}) &= U_1, \\
\pi_u(V_{c_2}) &= U_1 \oplus U_2, \\
&\vdots \\
\pi_u(V_{c_n}) &= U_1 \oplus \cdots \oplus U_n.
\end{aligned}$$

Then $U_1 \oplus \cdots \oplus U_n = U$, otherwise the representation $\boldsymbol{T}$ would contain indecomposable representations of the one-point set $\{a\}$ as direct summands.

Suppose that $u^i_1, \ldots u^i_{p_i}$ is a basis of $U_i$, and $e^i_j = u^i_j + f^i_j$ are those elements of the space $V_{c_i}$ for which $\pi_u(e^i_j) = u^i_j$ , where $f^i_j \in V_b$. Let's denote $F_i = \langle f^i_1, \ldots, f^i_{p_i} \rangle$. Since the complement $U$ is chosen modulo $\Delta = V_a \cap V_b$, and the spaces $U_i$ are chosen modulo $U_1 \oplus \cdots \oplus U_{i-1}$ and $f^i_j \in V_b$, then the elements $f^i_1, \ldots f^i_{p_i}$ can be chosen modulo the space $\Delta + (F_1 + \cdots + F_{i-1}) + (V_b \cap V_{c_i})$, and they must be linearly independent modulo this space, otherwise the representation $\boldsymbol{T}$ would contain as a direct summand an indecomposable representation of the set $\{a, c_i\}$.

Thus, we can set $\boldsymbol{T}' = \{W; W_x \mid x \in \mathcal{M}'_{(a,b)}\}$, where

$$\begin{aligned}
W &= V_b \oplus X \\
W_x &= W \cap V_x \text{ for } x \in \mathcal{M}'_{(a,b)} \setminus (C^+ \cup C^-) \\
W_{c_i^-} &= V_{c_i} \cap V_b \\
W_{c_i^+} &= (F_1 \oplus \cdots \oplus F_i) + (H_1 \oplus \cdots \oplus H_i) + \Delta + W_{c_i^-}.
\end{aligned}$$

In this case, obviously, $W_{c_i^+} = F_i \oplus H_i \oplus \sum_{x \prec c_i^+} W_x$ and, therefore, $[\boldsymbol{T}'] \simeq \boldsymbol{T}$, which was required to be proved.

For the purposes of [7], we will need another statement characterizing the operation of transition from the representation $\boldsymbol{S}$ of the set $\mathcal{M}'_{(a,b)}$ to the representation $[\boldsymbol{S}]$ of the set $\mathcal{M}$.

Let $S = \{A; A_x \mid x \in \mathcal{M}'_{(a,b)}\}$ be the representation of the derived set

$\mathcal{M}'_{(a,b)}$ over the ring of polynomials in one variable $k[X]$, i.e. a set of ring-generated $k[X]$–modules such that $A \supset A_x$ for for all $x$ and $A_x \subset A_y$ for $x \leq y$. Let $\varphi_i : A_{c_i^+} \longrightarrow A_{c_i^+}/\sum_{x<c_i^+} A_x$ be the natural projection and $\varphi_i(f_1^i), \ldots, \varphi_i(f_{m_i}^i)$ the minimal system of generators of the image. We can consider some free $k[X]$–module $U$ with basis $u_j^i$ $(1 \leq i \leq n; 1 \leq j \leq m_i)$ and, setting $B = U \oplus A; e_j^i = f_j^i + u_j^i$, obtain some representation $[S] = \{B; B_x \mid x \in \mathcal{M}\}$ of the set $\mathcal{M}$ over $k[X]$, where

$$
\begin{aligned}
B_x &= U \oplus A_x \text{ for } x \in \{a\}^{\triangle} ; \\
B_y &= A_y \text{ for } y \in \{b\}^{\triangledown} ; \\
B_{c_1} &= \{e_1^1, \ldots, e_{m_1}^1\} + A_{c_1^-} \text{ and} \\
B_{c_i} &= \{e_1^j, \ldots, e_{m_i}^j\} + A_{c_i^-} + B_{c_{i-1}} \text{ for } i \geq 2
\end{aligned}
$$

(of course, there is no connection with the differentiation operation here).

In the same way, we can obtain another representation $[S]$ of the set $\mathcal{M}$, assuming that some of $u_j^i$ are equal to 0. Since the number of elements $u_j^i$ forming $f_j^i$ is finite, we can equate to 0 only in a finite number of ways, then from this representation $[S]$ (with fixed $f_j^i$) we can obtain only a finite number of such representations $[S]$ of the set $M$. Let us number all these representations by elements of some finite set $I$ and write $[S]_i$, $i \in I$.

Let us agree on one more notation (see [8]). If $S = \{A; A_x \mid x \in \mathcal{M}\}$ is a representation of some set $\mathcal{M}$ over the polynomial ring $k[X]$ and $B$ is a finite-dimensional (over the field $k$) $k[X]$-module, then we denote by $S \otimes B$ the representation of the set $\mathcal{M}$ over $k$ , defined as follows: $S \otimes B = \{V; V_x \mid x \in \mathcal{M}\}$, where $V = A \otimes_{k[X]} B$ ; $V_x = Im\big(A_x \oplus_{k[X]} B \xrightarrow{\alpha_x \otimes 1} A \otimes_{k[X]} B\big)$ and $\alpha_x = A_x \longrightarrow A$ is the natural embedding.

The following follows directly from the definitions.

**Proposition 2.** Let $S$ be a representation of the derived set $\mathcal{M}'_{(a,b)}$ over a ring $k[X]$ and let B be a finite-dimensional $k[X]$–module. Then $[S \otimes B] \simeq [S]_i \otimes B$ for some $i \in I$.

In conclusion, we note that the algorithm for differentiating a partially ordered set described here is related to the algorithm of [1]: one differentiation in the sense of [1] is obtained as a composition of several differentiations in a new sense. Namely, let $\xi$ be the maximal point of some partially ordered set $\mathcal{M}$,

not included in the four elements that are incomparable with each other. Then, differentiating the set $\mathcal{M}$ in any way by suitable pairs of the form $(a, \xi)$, as long as possible, we obtain a set that is the ordinal sum (see [7]) of the derived set $\mathcal{M}'_\xi$ in the sense of [1] and the one-point set $\{\xi\}$.

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