

BIORTHOGONALITY IN BANACH SPACES AND THE P-IDEAL DICHOTOMY

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ABSTRACT. We analyse the impact of the P-ideal dichotomy on the existence of uncountable biorthogonal systems in nonseparable Banach spaces. Our main result states that under the P-ideal dichotomy, any nonseparable Banach space of density strictly smaller than $\mathfrak{b}$ with weak* sequentially compact dual unit ball has an uncountable biorthogonal system. As a consequence, we get that the following is equivalent to the cardinal assumption $\mathfrak{b} = \omega_2$ assuming PID: Every nonseparable Asplund space has an uncountable biorthogonal system.

1. INTRODUCTION

Given an uncountable set S , a P-ideal in S is a collection $\mathcal{I}$ of countable infinite subsets of S which form an ideal under inclusion and for every sequence $(I_n)_{n \in \omega}$ in $\mathcal{I}$ there is $I \in \mathcal{I}$ such that $I_n \setminus I$ is finite for every $n \in \omega$.

The second author introduced the following principle in [18]:

Definition 1 (P-Ideal Dichotomy). *For every P-ideal $\mathcal{I}$ of countable infinite subsets of some set S , either*

- (i) *there is an uncountable $X \subseteq S$ such that $[X]^\omega \subseteq \mathcal{I}$; or*
- (ii) *$S = \bigcup_{n \in \omega} S_n$ where for every $n \in \omega$ and any $I \in \mathcal{I}$, $|S_n \cap I| < \omega$.*

It was proved in [18] that the P-Ideal Dichotomy (or simply PID) follows from the Proper Forcing Axiom and is consistent with the Generalized Continuum Hypothesis (relative to the consistency of a supercompact cardinal). When the index set S is equal to ω_1 , the corresponding principle PID is consistent just relative to ZFC, see [1]. Many applications of PID have been obtained since it was first introduced, see [20]. In particular, several recent results assume PID to establish equivalences between cardinal inequalities of the form $\mathfrak{t} > \omega_1$, where $\mathfrak{t}$ is a characteristic of the continuum, and mathematical statements concerning objects of higher order (i.e., objects that are not necessarily reals), see e.g., [1, 3, 14, 18].

The present work follows this direction by establishing, under PID, the equivalence between one of the most basic cardinal assumptions of this kind and the

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existence of some uncountable structures in nonseparable Asplund spaces. We show that, under PID, every nonseparable Asplund space has an uncountable biorthogonal system if and only if $\mathfrak{b} > \omega_1$, where $\mathfrak{b}$ denotes the bounding number, i.e., the minimal cardinality of an unbounded family in ω^ω with respect to the eventual dominance preorder $\leq^*$. One of the important consequences of PID for this standard cardinal invariant of the continuum is that $\mathfrak{b} \in \{\omega_1, \omega_2\}$; see [20]. Hence, the following assertions are equivalent under PID:

- (a₁) $\mathfrak{b} > \omega_1$.
- (a₂) $\mathfrak{b} = \omega_2$.

Therefore, in all the equivalences obtained in this work, the cardinal inequality $\mathfrak{b} > \omega_1$ can, in fact, be replaced by the cardinal equality $\mathfrak{b} = \omega_2$. It would be interesting to determine whether this phenomenon also holds for other inequalities of the form $\mathfrak{t} > \omega_1$ that PID connects to standard problems in mathematics.

In a different direction, there has been considerable research on the existence of certain uncountable structures in nonseparable Banach spaces. Given $0 \leq \varepsilon < 1$, an ε -biorthogonal system in a Banach space X is a family $(x_\alpha, f_\alpha)_{\alpha \in \Gamma}$ of pairs in $X \times X^*$ such that $f_\alpha(x_\alpha) = 1$ and $|f_\alpha(x_\beta)| \leq \varepsilon$ for all $\alpha \neq \beta$ in Γ . It is a biorthogonal system in the particular case when $\varepsilon = 0$. A biorthogonal system $(x_\alpha, f_\alpha)_{\alpha \in \Gamma}$ is fundamental if $\overline{\text{span}}\{x_\alpha : \alpha \in \Gamma\} = X$.

It is well known that every infinite-dimensional Banach space X admits an infinite biorthogonal system and, if X is separable, a biorthogonal system $(x_n, f_n)_{n \in \omega}$ can be chosen to be fundamental and to satisfy $\overline{\text{span}}^{w^*}\{f_n : n \in \omega\} = X^*$.

Let us consider the following pairs of assertions:

- (b₁) Every Banach space of density ω_1 has an uncountable ε -biorthogonal system, for every $\varepsilon \in (0, 1)$.
- (b₂) Every nonseparable Banach space has an uncountable ε -biorthogonal system, for every $\varepsilon \in (0, 1)$.
- (c₁) Every Banach space of density ω_1 has an uncountable biorthogonal system.
- (c₂) Every nonseparable Banach space has an uncountable biorthogonal system.
- (d₁) Every Banach space of density ω_1 has a fundamental biorthogonal system.
- (d₂) Every Banach space of density ω_1 has a nonseparable quotient with a monotone Schauder basis.

Remark 2. $(b_1) \Leftrightarrow (b_2)$ and $(c_1) \Leftrightarrow (c_2)$ hold in ZFC because of Hahn-Banach Theorem. $(d_1) \Leftrightarrow (d_2)$ holds in ZFC thanks to a result by Plichko, [13] - see also [8, Theorem 4.15].

From now on, we use (b), (c), and (d) to refer to either of the two equivalent corresponding statements above. When working under PID, we also use (a) to refer to either of the assertions (a₁) and (a₂).

The main purpose of this paper is to prove the following result.

Theorem 3. *The following statements are equivalent assuming the P-ideal dichotomy:*

- (a) $\mathfrak{b} = \omega_2$.
- (b') Every nonseparable Asplund space has an uncountable ε -biorthogonal system, for every $\varepsilon \in (0, 1)$.

(c') *Every nonseparable Asplund space has an uncountable biorthogonal system.*

Notice that the statements (b') and (c') correspond to (b) and (c) restricted to Asplund spaces. Asplund spaces, also called strong differentiability spaces, are Banach spaces X such that every continuous convex function on X is Fréchet differentiable on a dense subset of X . This is equivalent to saying that every separable subspace of X has a separable dual. Since subspaces of Asplund spaces are Asplund, (b_1) (respectively, (c_1) and (d_1)) remains equivalent to (b_2) (respectively, (c_2) and (d_2)) when “Banach space” is replaced by “Asplund space”.

One interesting feature of our main result (Theorem 3) is that it connects phenomena from two different areas of mathematics, a connection that would have been difficult to anticipate without PID. However, this theorem also connects two known phenomena, (b') and (c'), from the geometry of Asplund spaces. In fact, the literature (see, e.g., [9, 11]) contains the question of whether (b) and (c) are equivalent conditions for a given Banach space. For example, it is shown in [9] that if a nonseparable Banach space X has an equivalent norm $\|\cdot\|'$ with the Mazur intersection property (which states that every closed convex subset of X is the intersection of closed $\|\cdot\|'$ -balls), then, for every $\varepsilon > 0$, the space X has an uncountable ε -biorthogonal system. The authors then ask whether one can also obtain an uncountable biorthogonal system. The motivation for this question is reinforced by another result from [9], which states that if a Banach space X has a biorthogonal system $(x_\alpha, f_\alpha)_{\alpha \in \Gamma}$ such that the functionals $(f_\alpha)_{\alpha \in \Gamma}$ span the dual space X^* , then X admits an equivalent renorming with the Mazur intersection property. On the other hand, it is shown in [11] that, in a forcing extension, there is an Asplund space X such that, for every $\varepsilon > 0$, there is an uncountable ε -biorthogonal system in X , but X has no uncountable biorthogonal system. Thus, some additional assumption, such as PID, is necessary for our Theorem 3.

Theorem 3 gives a partial answer to the following question, raised by the second author:

Question 4 (Todorćević, Question 26.7, [20]). *Under PID, are the following statements equivalent?*

- (a) $\mathfrak{b} > \omega_1$.
- (c) *Every nonseparable Banach space contains an uncountable biorthogonal system.*

This question was motivated, on the one hand, by the consistency result stating that every nonseparable Banach space has an uncountable biorthogonal system. This result was first obtained by the second author in [19] using Martin’s Maximum (MM). It was then reformulated in [20] by replacing MM with the conjunction of PID and one of the cardinal assumptions $\mathfrak{p} > \omega_1$ or $\mathfrak{m} > \omega_1$.

The maximal cardinality of a biorthogonal system in a given Banach space X , and its relation to the density of X , have been studied by several authors but are not yet completely understood. Since the 1980s, several consistent examples of nonseparable Banach spaces with no uncountable biorthogonal systems have been obtained under assumptions such as the Continuum Hypothesis or combinatorial principles such as $\diamond$ or $\clubsuit$; see [8]. Similar examples have also been obtained by forcing; see, for instance, [5, 11]. For a further discussion of this subject, see [10]. Among these examples, the one constructed by the second author in [16, Theorem

2.4] under the cardinal assumption $\mathfrak{b} = \omega_1$ is of particular importance to us here (see also [4]):

Theorem 5 (Todorćevic, [16], [17]). *Assuming $\mathfrak{b} = \omega_1$, there exists a compact Hausdorff scattered space K of weight ω_1 such that all finite powers K^n are hereditarily separable.*

The relevance of this result here can be briefly explained as follows. First, since the compact space K is scattered, the function space $C(K)$ is Asplund. The fact that all finite powers of K are hereditarily separable is equivalent to the fact that the function space $C(K)$ is hereditarily Lindelöf with respect to the weak topology and, in particular, has no ε -biorthogonal system for any $\varepsilon \geq 0$. All these facts are standard and can be found in the references given in the following sketch of the proof of our main theorem.

Sketch of the proof of Theorem 3: The implication $(c') \Rightarrow (b')$ is immediate. The contrapositive of $(b') \Rightarrow (a)$ follows from Theorem 5: the construction in [16, Theorem 2.4] yields, under $\mathfrak{b} = \omega_1$, a compact Hausdorff scattered space K of weight ω_1 whose finite powers K^n are all hereditarily separable. By a result of Namioka and Phelps [12], the fact that K is scattered implies that the Banach space $C(K)$ of all scalar-valued continuous functions on K , equipped with the supremum norm, is an Asplund space (of density ω_1). As mentioned above, the fact that K is scattered and has hereditarily separable finite powers implies that $C(K)$ is hereditarily Lindelöf with respect to the weak topology (see, e.g., [8, Theorem 4.38]). This, in turn, guarantees that $C(K)$ has no uncountable ε -biorthogonal systems for any $\varepsilon \in [0, 1)$. In particular, it has neither an uncountable biorthogonal system nor a fundamental biorthogonal system.

The rest of this paper is mostly devoted to prove that, under PID, $(a) \Rightarrow (c')$, to conclude the proof. $\square$

Notice that PID was not used in the previous sketch. The main role of PID can be illustrated by the following simple lemma, which provides typical useful examples of P-ideals under the assumption $\mathfrak{b} > \omega_1$.

Lemma 6. *If a family $\mathcal{F}$ of subsets of a set S is such that $|\mathcal{F}| < \mathfrak{b}$, then*

$$\mathcal{F}^\perp := \{I \in [S]^\omega : (\forall F \in \mathcal{F}) \quad |F \cap I| < \omega\}$$

is a P-ideal. $\square$

The proofs follow similar lines to those in [2, 19, 20], while weakening the cardinal assumption. In Section 2, we prove that $(a) \Rightarrow (c')$ under PID, and we also establish the crucial Theorem 7. In Section 3, we address the question of whether, under PID, (a) is equivalent to stronger statements, such as (d) for Asplund spaces. Our notation is fairly standard and follows [8].

2. UNCOUNTABLE BIORTHOGONAL SYSTEMS

Our main purpose in this section is to prove the following result.

Theorem 7. *Assume PID and $\mathfrak{b} > \omega_1$. If X is a Banach space with weak* sequentially compact dual unit ball which contains a dense $\mathbb{Q}$ -linear subspace D of*

cardinality ω_1 , then there exists an uncountable bounded sequence $(f_\alpha : \alpha < \omega_1)$ of (distinct) functionals in X^* such that

$$(1) \quad \forall x \in X \quad (f_\alpha(x) : \alpha < \omega_1) \in c_0(\omega_1)$$

and

$$(2) \quad \forall x \in D \quad (f_\alpha(x) : \alpha < \omega_1) \in \ell_1(\omega_1).$$

This theorem is crucial for the subsequent construction of biorthogonal systems, whose functionals will be obtained by refining the sequence of functionals satisfying properties (1) and (2) above. For instance, we readily obtain the following corollary, which completes the proof that (a) implies (c'):

Corollary 8. *Assume PID and $\mathfrak{b} > \omega_1$. Every nonseparable Asplund space X contains an uncountable biorthogonal system.*

Proof. Stegall showed in [15] that Asplund spaces have weak* sequentially compact dual unit balls. Hence, given an Asplund space X of density ω_1 , we can apply Theorem 7, and get for a dense $\mathbb{Q}$ -linear subspace D of X of cardinality ω_1 , a bounded sequence $(f_\alpha)_{\alpha \in \omega_1}$ of (distinct) functionals in X^* satisfying properties (1) and (2). For each $\alpha \in \omega_1$, let $x_\alpha \in D$ be such that $f_\alpha(x_\alpha) \neq 0$. It follows from [7, Proposition 11] that X has an uncountable biorthogonal system. $\square$

Property (2) clearly implies property (1). We stated Theorem 11 in this way because its proof proceeds in two steps: first, we establish property (1), and then we establish property (2). We first need the following elementary lemma.

Lemma 9. *Let K be a topological Hausdorff space of weight strictly less than $\mathfrak{b}$. If a sequence $(x_n)_{n \in \omega} \subseteq K$ converges to some $x \in K$ and, for each $n \in \omega$, $(y_k^n)_{k \in \omega} \subseteq K$ is a sequence converging to x_n , then there is an increasing sequence $(k_n)_{n \in \omega}$ such that $(y_{k_n}^n)_{n \in \omega}$ converges to x . Consequently, the sequential closure $\text{Seq}(S)$ of any set $S \subseteq K$ is the set of all limits of sequences of elements of S .* $\square$

The following proposition guarantees the existence of a sequence of functionals satisfying property (1):

Proposition 10. *Assume PID and $\mathfrak{b} > \omega_1$. If X is a Banach space of density ω_1 whose dual unit ball is weak* sequentially compact, then there is an uncountable bounded sequence $(f_\alpha)_{\alpha \in \omega_1}$ of (distinct) functionals in X^* such that*

$$(1) \quad \forall x \in X \quad (f_\alpha(x) : \alpha < \omega_1) \in c_0(\omega_1)$$

Proof. Fix D a dense $\mathbb{Q}$ -linear subspace of X of cardinality ω_1 . It is easy to construct an increasing chain $(X_\alpha)_{\alpha < \omega_1}$ of separable subspaces of X such that $\dim(X_{\alpha+1}/X_\alpha) = 1$ for $\alpha < \omega_1$, if $\lambda < \omega_1$ is a limit ordinal, then $X_\lambda = \overline{\text{span}} \bigcup_{\alpha < \lambda} X_\alpha$ and $X = \overline{\text{span}} \bigcup_{\alpha < \omega_1} X_\alpha$. Using Hahn-Banach Theorem, we can obtain a normalized sequence $(h_\alpha)_{\alpha \in \omega_1}$ of linearly independent functionals in X^* such that for every $x \in D$, $h_\alpha(x) = 0$ for all but countably many $\alpha \in \omega_1$.

Let

$$\mathcal{I}_1 = \{I \in [\omega_1]^\omega : (\forall x \in D)(\forall \varepsilon \in (0, 1) \cap \mathbb{Q}) \quad \{\alpha \in I : |h_\alpha(x)| \geq \varepsilon\} \text{ is finite}\}$$

and by Lemma 6, $\mathcal{I}_1$ is a P-ideal. Apply PID and notice that if alternative (i) holds, we are done. If alternative (ii) of PID holds, let $\Gamma \in [\omega_1]^{\omega_1}$ be such that $|I \cap \Gamma| < \omega$ for every $I \in \mathcal{I}_1$. Hence $0 \notin \text{Seq}(\{h_\alpha : \alpha \in \Gamma\})$. Indeed, if $0 \in \text{Seq}(\{h_\alpha : \alpha \in \Gamma\})$, then by Lemma 9 there is $(\alpha_n)_{n \in \omega} \subseteq \Gamma$ such that $(h_{\alpha_n})_{n \in \omega}$ is weak* convergent to

0, so that $I = \{\alpha_n : n \in \omega\} \in \mathcal{I}_1$. This contradicts the fact that $|I \cap \Gamma| < \omega$ for every $I \in \mathcal{I}_1$. On the other hand, $0 \in \overline{\{h_\alpha : \alpha \in \Gamma\}}^{w^*}$.

Consider $K = \text{Seq}(\{h_\alpha : \alpha \in \Gamma\})$, $S = \{g - f : f \neq g \text{ and } f, g \in K\}$ and

$$\mathcal{I}_2 = \{I \in [S]^\omega : (\forall x \in D)(\forall \varepsilon \in (0, 1) \cap \mathbb{Q}) \quad \{h \in I : |h(x)| \geq \varepsilon\} \text{ is finite}\}.$$

Again, by Lemma 6, $\mathcal{I}_2$ is a P-ideal. We shall prove that alternative (ii) of PID cannot hold.

Given a weak*-open neighborhood V of 0, since $0 \in \overline{\{h_\alpha : \alpha \in \Gamma\}}^{w^*}$, we know that $V \cap \{h_\alpha : \alpha \in \Gamma\}$ is infinite, so that its sequential closure $\text{Seq}(V \cap \{h_\alpha : \alpha \in \Gamma\})$ is a weak* sequentially closed subset of K which doesn't contain 0.

Let $\mathcal{F}$ be a maximal filter of weak* sequentially closed subsets of K containing $\text{Seq}(V \cap \{h_\alpha : \alpha \in \Gamma\})$ for all weak*-open neighborhood V of 0. Using the weak* sequential compactness of the dual unit ball and the maximality of $\mathcal{F}$, we obtain that $\mathcal{F}$ is σ -complete, in the sense that the intersection of countably many elements of $\mathcal{F}$ belongs to $\mathcal{F}$. Let $\mathcal{F}^+$ be the co-ideal of positive subsets of K , that is, the family of all subsets Z of K that intersect every element of $\mathcal{F}$. Observe that if $Z \in \mathcal{F}^+$, then $\text{Seq}(Z) \in \mathcal{F}$. Also, if $\bigcup_{n \in \omega} Z_n \in \mathcal{F}^+$, then there is $n \in \omega$ such that $Z_n \in \mathcal{F}^+$. A Fubini-type argument will give the following claim:

Claim 10.1. *If $S = \bigcup_{n \in \omega} S_n$, then there is $n_0 \in \omega$ and a set $Z \in \mathcal{F}^+$ such that*

$$(\forall f \in Z) \quad Z_f = \{g \in K : g - f \in S_{n_0}\} \in \mathcal{F}^+.$$

Proof of the claim. For each $n \in \omega$, let

$$Y_n = \{f \in K : \{g \in K : g - f \in S_n\} \in \mathcal{F}^+\}.$$

Notice that for any fixed $f \in K$,

$$\bigcup_{n \in \omega} \{g \in K : g - f \in S_n\} = \{g \in K : g - f \in \bigcup_{n \in \omega} S_n\} = \{g \in K : g - f \in S\} = K.$$

Therefore,

$$\bigcup_{n \in \omega} Y_n = \{f \in K : \exists n \in \omega \text{ such that } \{g \in K : g - f \in S_n\} \in \mathcal{F}^+\} = K,$$

Now, suppose by contradiction that, for every $n \in \omega$, $Y_n \notin \mathcal{F}^+$. Then, for each $n \in \omega$, there is $F_n \in \mathcal{F}$ such that $F_n \cap Y_n = \emptyset$. Therefore,

$$\left(\bigcap_{n \in \omega} F_n\right) \cap \left(\bigcup_{n \in \omega} Y_n\right) = \emptyset.$$

Since $\bigcap_{n \in \omega} F_n \in \mathcal{F}$, we get that $K = \bigcup_{n \in \omega} Y_n \notin \mathcal{F}^+$, a contradiction. $\square$

We are now ready to prove the following claim, which ensures that alternative (ii) of PID cannot hold for $\mathcal{I}_2$:

Claim 10.2. *If $S = \bigcup_{n \in \omega} S_n$, then there is $n_0 \in \omega$ such that $0 \in \text{Seq}(S_{n_0})$.*

Proof of the claim. If $S = \bigcup_{n \in \omega} S_n$, Claim 10.1 gives an $n_0 \in \omega$ and a set $Z \in \mathcal{F}^+$ such that

$$(\forall f \in Z) \quad Z_f = \{g \in K : g - f \in S_{n_0}\} \in \mathcal{F}^+.$$

Fix $f_0 \in \text{Seq}(Z)$. Inductively, given

$$f_n \in \text{Seq}(Z) \cap \left(\bigcap_{k \in \omega, i < n} \text{Seq}(Z_{f_k^i})\right),$$

use Lemma 9 to get $(f_k^n)_{k \in \omega} \subseteq Z$ such that $f_n = w^* - \lim_k f_k^n$. Since by our hypothesis, for each $k \in \omega$, $Z_{f_k^n} \in \mathcal{F}^+$, we get that

$$\text{Seq}(Z) \cap \left(\bigcap \{ \text{Seq}(Z_{f_k^i}) : k \in \omega, i < n+1 \} \right) \in \mathcal{F},$$

so that we can fix

$$f_{n+1} \in \text{Seq}(Z) \cap \left(\bigcap \{ \text{Seq}(Z_{f_k^i}) : k \in \omega, i < n+1 \} \right).$$

Now we have a sequence $(f_n)_{n \in \omega} \subseteq \text{Seq}(Z)$ and by the hypothesis that B_{X^*} is weak* sequentially compact, we may assume without loss of generality (passing to a subsequence) that $(f_n)_{n \in \omega}$ converges weakly* to some f , so that $(f_{2n} - f_{2n+1})_{n \in \omega}$ converges weakly* to 0.

For each $n \in \omega$, since $f_{2n+1} \in \bigcap \{ \text{Seq}(Z_{f_k^{2n}}) : k \in \omega \}$, let for each $k \in \omega$ $(g_i^{n,k})_{i \in \omega} \subseteq Z_{f_k^{2n}}$ be such that $f_{2n+1} = w^* - \lim_i g_i^{n,k}$ for every $k \in \omega$, again by Lemma 9, and for each $n \in \omega$, there is an increasing sequence $(i_k)_{k \in \omega}$ such that $f_{2n+1} = w^* - \lim_k g_{i_k}^{n,k}$, so that $(f_k^{2n} - g_{i_k}^{n,k})_{k \in \omega}$ converges weakly* to $f_{2n} - f_{2n+1}$. Applying Lemma 9 once again, there is an increasing sequence $(k_n)_{n \in \omega}$ such that $(f_{k_n}^{2n} - g_{i_{k_n}}^{n,k_n})_{n \in \omega}$ converges weakly* to 0. On the other hand, let us go back to the choice of these sequences to see that $(f_{k_n}^{2n} - g_{i_{k_n}}^{n,k_n})_{n \in \omega} \subseteq S_{n_0}$ since $g_{i_{k_n}}^{n,k_n} \in Z_{f_{k_n}^{2n}}$. This concludes the proof of the claim. $\square$

The previous claim guarantees that alternative (ii) of PID cannot hold for $\mathcal{I}_2$. Hence, alternative (i) holds. Thus, we obtain an uncountable family of functionals $(f_\alpha)_{\alpha \in \omega_1}$ such that, for every $x \in D$ and, consequently, for every $x \in X$, $(f_\alpha(x))_{\alpha \in \omega_1} \in c_0(\omega_1)$. This concludes the proof of the proposition. $\square$

We are now ready to prove the next step in the main result of this section.

Proposition 11. *Assume PID and $\mathfrak{b} > \omega_1$. Let X be a Banach space of density ω_1 , let D be a dense $\mathbb{Q}$ -linear subspace of X of cardinality ω_1 , and let $(f_\alpha : \alpha < \omega_1)$ be an uncountable bounded sequence of (distinct) functionals in X^* such that*

$$(1) \quad \forall x \in X \quad (f_\alpha(x) : \alpha < \omega_1) \in c_0(\omega_1)$$

Then there exists $\Gamma \in [\omega_1]^{\omega_1}$ such that

$$(2) \quad \forall x \in D \quad (f_\alpha(x) : \alpha < \omega_1) \in \ell_1(\Gamma).$$

Proof. Let

$$\mathcal{I} = \{ I \in [\omega_1]^\omega : (\forall x \in D) \sum_{\alpha \in I} |f_\alpha(x)| < +\infty \}.$$

Let us prove that $\mathcal{I}$ is a P-ideal.

Given a pairwise disjoint sequence $(I_n)_{n \in \omega}$ of elements of $\mathcal{I}$, fix an enumeration e of $\bigcup_{n \in \omega} I_n$ and for each $x \in D$, let $g_x \in \omega^\omega$ be such that

$$\sum_{\alpha \in I_n \setminus e[g_x(n)]} |f_\alpha(x)| < \frac{1}{2^n}.$$

Since $\mathfrak{b} > \omega_1$, there is $g \in \omega^\omega$ such that $g_x \leq^* g$ for every $x \in D$. Let

$$I = \bigcup_{n \in \omega} I_n \setminus e[g(n)].$$

Clearly $I_n \subseteq^* I$ for every $n \in \omega$ and let us prove that $I \in \mathcal{I}$. Given $x \in D$, let $n_0 \in \omega$ be such that $n \geq n_0$ implies that $g_x(n) \leq g(n)$. Then, for each $n < n_0$,

$$\sum_{\alpha \in I_n \setminus e[g(n)]} |f_\alpha(x)| \leq \sum_{\alpha \in I_n} |f_\alpha(x)| < +\infty$$

and for $n \geq n_0$,

$$\sum_{\alpha \in I_n \setminus e[g(n)]} |f_\alpha(x)| \leq \sum_{\alpha \in I_n \setminus e[g_x(n)]} |f_\alpha(x)| < \frac{1}{2^n}.$$

Hence,

$$\begin{aligned} \sum_{\alpha \in I} |f_\alpha(x)| &= \sum_{n \in \omega} \sum_{\alpha \in I_n \setminus e[g(n)]} |f_\alpha(x)| \\ &= \sum_{n < n_0} \sum_{\alpha \in I_n \setminus e[g(n)]} |f_\alpha(x)| + \sum_{n \geq n_0} \sum_{\alpha \in I_n \setminus e[g(n)]} |f_\alpha(x)| < +\infty, \end{aligned}$$

so that $I \in \mathcal{I}$, which concludes the proof that $\mathcal{I}$ is a P-ideal.

Apply PID and notice that if alternative (i) holds, we are done. Let us see that alternative (ii) of PID cannot hold, because any infinite subset of ω_1 contains an infinite subset which is in $\mathcal{I}$. Let $I \in [\omega_1]^\omega$ and fix an enumeration e of I . For each $x \in D$, let $h_x \in \omega^\omega$ be such that for every $n \geq h_x(k)$, $|f_{e(n)}(x)| < \frac{1}{2^k}$.

Since $\mathfrak{b} > \omega_1$, there is $h \in \omega^\omega$ such that $h_x \leq^* h$ for every $x \in D$. Without loss of generality we may assume that h is strictly increasing. Let $B = \{e(h(n)) : n \in \omega\} \in [I]^\omega$ and let us see that $B \in \mathcal{I}$: given $x \in D$, let $n_0 \in \omega$ be such that $n \geq n_0$ implies $h_x(n) \leq h(n)$, so that $|f_{e(h(n))}(x)| < \frac{1}{2^n}$. Hence,

$$\begin{aligned} \sum_{\alpha \in B} |f_\alpha(x)| &= \sum_{n \in \omega} |f_{e(h(n))}(x)| = \sum_{n < n_0} |f_{e(h(n))}(x)| + \sum_{n \geq n_0} |f_{e(h(n))}(x)| \\ &\leq \sum_{n < n_0} |f_{e(h(n))}(x)| + \sum_{n \geq n_0} \frac{1}{2^n} < +\infty. \end{aligned}$$

This guarantees that B belongs to the ideal $\mathcal{I}$, so that (ii) of PID cannot happen and this concludes the proof. $\square$

Proof of Theorem 7. Since the dual unit ball of an Asplund space is weak* sequentially compact (see [6]), we can apply Proposition 10 to get an uncountable bounded sequence $(f_\alpha : \alpha < \omega_1)$ of functionals in X^* satisfying (1). This sequence can be then refined by Proposition 11, which yields the required sequence satisfying (1) and (2). $\square$

3. NONSEPARABLE QUOTIENT

This section is motivated by the following natural question:

Question 12. *Under PID, does $\mathfrak{b} > \omega_1$ imply that:*

- (d') *Every Asplund space of density ω_1 has a nonseparable quotient with a monotone Schauder basis.*

We prove the following result:

Proposition 13. *Let X be a Banach space with weak* sequentially compact dual unit ball which contains a dense $\mathbb{Q}$ -linear subspace D of cardinality ω_1 and let $(f_\alpha : \alpha < \omega_1)$ be an uncountable bounded sequence $(f_\alpha)_{\alpha < \omega_1}$ of (distinct) functionals in X^* such that*

$$(2) \quad (\forall x \in D) \quad (f_\alpha(x) : \alpha < \omega_1) \in \ell_1(\omega_1).$$

Then there exists $\Gamma \in [\omega_1]^{\omega_1}$ such that $(f_\alpha)_{\alpha \in \Gamma}$ is a monotone long Schauder basic sequence in X^ and $Q : X \rightarrow (\overline{\text{span}}\{f_\alpha : \alpha \in \Gamma\})^*$, defined by*

$$Q(x)(f) = f(x)$$

is a bounded linear operator such that

$$\text{span}\{f_\alpha^* : \alpha \in \Gamma\} \subseteq Q[X] \subseteq \overline{\text{span}}\{f_\alpha^* : \alpha \in \Gamma\}.$$

Proof of Proposition 13. Starting from a sequence $(f_\alpha : \alpha < \omega_1)$ as in the hypothesis, we can apply [7, Proposition 11] and get $\Gamma_0 \in [\omega_1]^{\omega_1}$ and $(x_\alpha)_{\alpha \in \Gamma_0}$ in X such that $(x_\alpha, f_\alpha)_{\alpha \in \Gamma_0}$ is a biorthogonal system.

Let us now find an uncountable $\Gamma \subseteq \Gamma_0$ such that $(f_\alpha)_{\alpha \in \Gamma}$ is a long monotone basic sequence.

Claim 13.1. *For every $\delta, \varepsilon \in (0, 1)$ and every $\Delta \in [\Gamma_0]^{<\omega}$ there exists $\Lambda \in [\Gamma_0]^{<\omega}$ such that for every norm-one $f^* \in (\text{span}\{f_\alpha : \alpha \in \Delta\})^*$ there is a norm-one $x \in X$ such that:*

- (i) $|f^*(e) - e(x)| \leq \delta \cdot \|e\|$ for all $e \in \text{span}\{f_\alpha : \alpha \in \Delta\}$;
- (ii) $\sum_{\alpha \in \omega_1 \setminus \Lambda} |f_\alpha(x)| < \varepsilon$.

Proof of the claim. Since $G := \text{span}\{f_\alpha : \alpha \in \Delta\} \subseteq X^*$ is finite-dimensional, the sphere of $(\text{span}\{f_\alpha : \alpha \in \Delta\})^*$ contains a finite $\frac{\delta}{5}$ -dense subset D^* .

Fix $g^* \in D^*$. By the principle of local reflexivity, there is $y_{g^*} \in X$ with $\|g^*\| - \|y_{g^*}\| < \frac{\delta}{5}$ such that $|g^*(e) - e(y_{g^*})| \leq \frac{\delta}{5} \cdot \|e\|$ for all $e \in G$. Let $x_{g^*} \in D$ be such that $\|y_{g^*} - x_{g^*}\| < \frac{\delta}{5}$. Given $\varepsilon > 0$, since $x_{g^*} \in D$, there is $\Gamma_{g^*} \in [\omega_1]^{<\omega}$ such that

$$\sum_{\alpha \in \omega_1 \setminus \Gamma_{g^*}} |f_\alpha(x_{g^*})| < \varepsilon \cdot \|x_{g^*}\|.$$

Now let

$$\Lambda = \bigcup \{\Gamma_{g^*} : g^* \in D^*\},$$

which is clearly finite as a finite union of finite sets. Let us show that Λ has the required properties. Given a norm-one $f^* \in (\text{span}\{f_\alpha : \alpha \in \Delta\})^*$, let $g^* \in D^*$ be such that $\|f^* - g^*\| < \frac{\delta}{5}$ and let $x = \frac{x_{g^*}}{\|x_{g^*}\|}$.

To prove (i), first notice that $\|y_{g^*}\| \in (1 - \frac{\delta}{5}, 1 + \frac{\delta}{5})$, so that $\|x_{g^*}\| \in (1 - \frac{2\delta}{5}, 1 + \frac{2\delta}{5})$. Therefore,

$$\|x_{g^*} - x\| = \left\| x_{g^*} - \frac{x_{g^*}}{\|x_{g^*}\|} \right\| = \left| \|x_{g^*}\| - 1 \right| < \frac{2\delta}{5}.$$

Hence, for every $e \in G$ we have that

$$\begin{aligned} |f^*(e) - e(x)| &\leq |f^*(e) - g^*(e)| + |g^*(e) - e(y_{g^*})| + |e(y_{g^*}) - e(x_{g^*})| + |e(x_{g^*}) - e(x)| \\ &\leq \|f^* - g^*\| \cdot \|e\| + \frac{\delta}{5} \cdot \|e\| + \|y_{g^*} - x_{g^*}\| \cdot \|e\| + \|x_{g^*} - x\| \cdot \|e\| < \delta \cdot \|e\|, \end{aligned}$$

which concludes the proof of (i).

Since $\Gamma_{g^*} \subseteq \Lambda$, clearly we get that

$$\sum_{\alpha \in \omega_1 \setminus \Lambda} |f_\alpha(x)| = \frac{1}{\|x_{g^*}\|} \sum_{\alpha \in \omega_1 \setminus \Lambda} |f_\alpha(x_{g^*})| \leq \frac{1}{\|x_{g^*}\|} \sum_{\alpha \in \omega_1 \setminus \Gamma_{g^*}} |f_\alpha(x_{g^*})| < \varepsilon,$$

which ensures (ii) and concludes the proof of Claim 13.1. $\square$

Let Ω be the set of limit ordinals $\lambda < \omega_1$ with the following property: for every $\delta, \varepsilon \in (0, 1)$, every $\Delta \in [\Gamma_0 \cap \lambda]^{<\omega}$ and every norm-one $f^* \in (\text{span}\{f_\alpha : \alpha \in \Delta\})^*$ there is a norm-one $x \in X$ such that $|f^*(e) - e(x)| \leq \delta \cdot \|e\|$ for all $e \in \text{span}\{f_\alpha : \alpha \in \Delta\}$ and $\sum_{\alpha \in \omega_1 \setminus \lambda} |f_\alpha(x)| < \varepsilon$. Let us show that Ω is unbounded in ω_1 .

To prove that Ω is unbounded, we first fix, for each $\delta, \varepsilon \in \mathbb{Q} \cap (0, 1)$ and each $\Delta \in [\Gamma_0]^{<\omega}$, a finite subset $\Lambda_{\delta, \varepsilon, \Delta}$ of Γ_0 as in Claim 13.1.

Given $\mu < \omega_1$, let $\lambda_0 = \mu$ and recursively construct an increasing sequence $(\lambda_n)_{n \in \omega}$ in ω_1 as follows: given λ_n , let $\lambda_{n+1} > \lambda_n$ be a countable ordinal such that

$$\bigcup \{\Lambda_{\delta, \varepsilon, \Delta} : \delta, \varepsilon \in \mathbb{Q} \cap (0, 1), \Delta \in [\Gamma_0 \cap \lambda_n]^{<\omega}\} \subseteq \lambda_{n+1}.$$

Notice that this is possible since we have countably many triples $(\delta, \varepsilon, \Delta)$, each determining a finite set $\Lambda_{\delta, \varepsilon, \Delta} \subseteq \omega_1$.

Let $\lambda = \sup_{n \in \omega} \lambda_n$ and clearly $\lambda > \mu$. To prove that $\lambda \in \Omega$, let $\delta, \varepsilon \in (0, 1)$ and $\Delta \in [\Gamma_0 \cap \lambda]^{<\omega}$. Let $\delta', \varepsilon' \in \mathbb{Q} \cap (0, 1)$ be such that $\delta' \leq \delta$ and $\varepsilon' \leq \varepsilon$. Also, let $n \in \omega$ be such that $\Delta \subseteq \lambda_n$. It follows from the construction that $\Lambda_{\delta', \varepsilon', \Delta} \subseteq \lambda_{n+1} \subseteq \lambda$. Then, the properties of $\Lambda_{\delta', \varepsilon', \Delta}$ (obtained from Claim 13.1) guarantee that for every norm-one $f^* \in (\text{span}\{f_\alpha : \alpha \in \Delta\})^*$, there is a norm-one $x \in X$ such that

$$|f^*(e) - e(x)| \leq \delta' \cdot \|e\| \leq \delta \cdot \|e\|$$

for all $e \in \text{span}\{f_\alpha : \alpha \in \Delta\}$ and

$$\sum_{\alpha \in \omega_1 \setminus \lambda} |f_\alpha(x)| \leq \sum_{\alpha \in \omega_1 \setminus \Lambda_{\delta', \varepsilon', \Delta}} |f_\alpha(x)| < \varepsilon' \leq \varepsilon.$$

Therefore, $\lambda \in \Omega$, which ensures that Ω is unbounded.

Finally, let $\Gamma \in [\Gamma_0]^{\omega_1}$ be such that for every $\mu < \lambda$ consecutive elements of Ω there is at most one element of Γ .

Claim 13.2. $(f_\alpha)_{\alpha \in \Gamma}$ is a long monotone Schauder basic sequence.

Proof of Claim 13.2. Given $g = \sum_{i=1}^n a_i f_{\alpha_i}$ and $f = \sum_{i=1}^m a_i f_{\alpha_i}$ for some $\alpha_1 < \dots < \alpha_n$ in Γ , $a_1, \dots, a_n \in \mathbb{R}$ and $1 \leq m < n$, we prove that $\|f\| \leq \|g\|$.

Suppose by contradiction that there is $\theta > 1$ such that $\|f\| > \theta \|g\|$. We may assume without loss of generality that $\|f\| = 1$. Let $\lambda \in \Omega$ be such that $\alpha_m < \lambda \leq \alpha_{m+1}$, $\Delta = \{\alpha_1, \dots, \alpha_m\}$ and a norm-one $f^* \in (\text{span}\{f_\alpha : \alpha \in \Delta\})^*$ such that $|f^*(f)| = \|f\| = 1$. Given $\delta \in (0, 1)$, let $\varepsilon \in (0, 1)$ be such that $\max\{|a_i| : 1 \leq i \leq n\} \cdot \varepsilon < \delta$. Since $\Delta \in [\omega_1 \cap \lambda]^{<\omega}$ and $\lambda \in \Omega$, there is a norm-one $x \in X$ such that $|f^*(e) - e(x)| \leq \delta \cdot \|e\|$ for all $e \in \text{span}\{f_\alpha : \alpha \in \Delta\}$ and $\sum_{\alpha \in \omega_1 \setminus \lambda} |f_\alpha(x)| < \varepsilon$. In particular, $|1 - f(x)| = |f^*(f) - f(x)| \leq \delta$ and $\sum_{i=m+1}^n |f_{\alpha_i}(x)| < \varepsilon$, so that

$$\sum_{i=m+1}^n |a_i f_{\alpha_i}(x)| \leq \max\{|a_i| : m+1 \leq i \leq n\} \cdot \varepsilon < \delta.$$

Then we get that

$$1 = \|f\| > \theta \|g\| \geq \theta |g(x)| \geq \theta (|f(x)| - \sum_{i=m+1}^n |a_i f_{\alpha_i}(x)|) \geq \theta(1 - \delta).$$

For a sufficiently small δ , we get a contradiction. $\square$

We are now ready to consider the map $Q : X \rightarrow (\overline{\text{span}}\{f_\alpha : \alpha \in \Gamma\})^*$, defined by

$$Q(x)(f) = f(x).$$

Q is clearly a bounded linear operator. On one hand, for every $\beta \in \Gamma$ and any $f = \sum_{\alpha \in \Gamma} f_\alpha^*(f) f_\alpha \in \overline{\text{span}}\{f_\alpha : \alpha \in \Gamma\}$, we have that

$$Q(x_\beta)(f) = \sum_{\alpha \in \Gamma} f_\alpha^*(f) f_\alpha(x_\beta) = f_\beta^*(f),$$

so that $\text{span}\{f_\alpha^* : \alpha \in \Gamma\} \subseteq Q[X]$.

On the other hand, for every $x \in D$, $\sum_{\alpha \in \Gamma} |f_\alpha(x)| < \infty$ and therefore,

$$\sum_{\alpha \in \Gamma} f_\alpha(x) f_\alpha^* \in \overline{\text{span}}\{f_\alpha^* : \alpha \in \Gamma\}.$$

This implies that $Q(x) = \sum_{\alpha \in \Gamma} f_\alpha(x) f_\alpha^*$, so that

$$\text{span}\{f_\alpha^* : \alpha \in \Gamma\} \subseteq Q[X] \subseteq \overline{\text{span}}\{f_\alpha^* : \alpha \in \Gamma\},$$

which concludes the proof. $\square$

At this point, we do not know whether the uncountable set Γ can be chosen so that

$$Q[X] = \overline{\text{span}}\{f_\alpha^* : \alpha \in \Gamma\}.$$

If this holds, it would give a positive answer to Question 12 and add (d') to the list of statements that are equivalent to $\mathfrak{b} > \omega_1$ under PID.

Remark 14. *If the answer to this question is positive, then, inspired by the main result in [2], we expect that, with some additional work, it could be shown that the dual of every Asplund space of density ω_1 has a fundamental biorthogonal system whose functionals are indeed in the space X . If this is the case, then, by the results of [9, Theorem 2.4], we could add yet another statement to our list of equivalences of $\mathfrak{b} = \omega_2$ under PID: Every Asplund space X of density ω_1 has a renorming with the Mazur intersection property, that is, a renorming such that every closed convex subset of X is the intersection of closed balls in X .*

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