ISOMETRIES OF JAMES-SCHREIER AND LORENTZ SPACES

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Abstract. We study the group of surjective linear isometries on certain real Banach sequence spaces using the preservation of extreme points in the closed unit ball. Our main result provides a characterization of the extreme points of the dual unit ball of the James-Schreier space V_1. As a consequence, we show that the only isometries on V_1 are $\pm Id$. We also obtain a Banach-Stone-type result for Lorentz sequence spaces, analogous to one proved in [13] for Lorentz function spaces.

1. Introduction

The study of surjective isometries on infinite-dimensional Banach spaces goes back to Banach's 1932 monograph [4], in which he characterized these maps on classical spaces such as $C[0,1]$ and L_p for $p \geq 1$. He further asserted that similar descriptions hold for c_0 and ℓ_p, without including proofs. In this paper, we focus our attention on two types of real Banach sequence spaces: James-Schreier spaces and Lorentz spaces. Our main goal is to characterize their surjective isometries in terms of permutations of the canonical basis and sign changes.

Given a real Banach space X, let us denote by $\mathrm{Isom}(X)$ the group of all surjective linear isometries $T : X \to X$. We say that X has a *trivial* group of isometries if every isometry is a multiple of the identity map Id, i.e., $\mathrm{Isom}(X) = \{\pm Id\}$. If X is a sequence space, following [1], we say that the group of isometries of X is *diagonal* if it consists of all operators $T : X \to X$ of the form:

$$T(x_n) = (\varepsilon_n x_n)$$

for some sequence of signs $(\varepsilon_n) \in \{-1,1\}^{\mathbb{N}}$. In this case, $\mathrm{Isom}(X) \cong \mathbb{Z}_2^{\mathbb{N}}$. Finally, $\mathrm{Isom}(X)$ is *standard* if it consists of all operators $T : X \to X$ of the form:

$$T(x_n) = (\varepsilon_n x_{\pi(n)})$$

for some $(\varepsilon_n) \in \{-1,1\}^{\mathbb{N}}$ and some $\pi \in S_\infty$, where S_∞ denotes the set of all permutations of the natural numbers. Analogous definitions can be considered in the complex case, by replacing $\{-1,1\}$ by the unit circle in $\mathbb{C}$, but for simplicity, we restrict our study to real Banach spaces.

Although every separable real Banach space admits an equivalent norm with trivial isometry group [5], certain spaces already exhibit this rigidity under their natural norm. In particular, the James space J_2 has trivial isometry group [6], and this was later extended to J_p, $p > 1$, [25]. The notions of diagonal and standard

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isometry groups, on the other hand, depend on the chosen Schauder basis used to represent the space as a sequence space. We omit explicit reference to the basis whenever the canonical one is used.

Between the trivial and the standard cases lie spaces with intermediate rigidity. Generalized Schreier spaces and their p-convexifications $X^p_{\mathcal{S}_\alpha}$, for $p \geq 1$, have diagonal isometry group [2, 11, 15]. More general families yield further restrictions on admissible permutations that characterize their isometry groups [12]. These results rely on two main ingredients: a description of the extreme points of the dual unit ball and the preservation of disjoint supports by isometries [3, 6, 11].

Our main result, Theorem 3.4, characterizes the extreme points of the dual unit ball of the James-Schreier space V_1. The spaces V_p, $p \geq 1$, introduced in [8, 7, 9], combine features of the James space J_p and the Schreier space in a way analogous to the construction of the combinatorial spaces $X^p_{\mathcal{F}}$. Using the fact that the canonical basis of V_1 is shrinking together with a detailed analysis of its unit ball, we describe the extreme points of $B_{V_1^*}$ in terms of their supports and coordinates. As a consequence, we prove in Theorem 4.4 that V_1 has trivial isometry group. Thus, although V_1 is built from James and Schreier structures, it exhibits isometric behavior markedly different from that of either of its components.

The study of surjective linear isometries can be framed in a Banach–Stone-type perspective: within a given class of Banach spaces, when does an isometry between two spaces force an equivalence of their underlying parameters? The classical Banach–Stone Theorem asserts that $C(K)$ and $C(L)$ are isometrically isomorphic if and only if K and L are homeomorphic. Likewise, ℓ_p and ℓ_q are isometrically isomorphic if and only if $p = q$. Analogous results hold for the p-convexified Schreier spaces $X^p_{\mathcal{S}}$ and the James–Schreier spaces V_p (see [9]), as well as for more general combinatorial spaces $X_{\mathcal{F}}$ and $X^p_{\mathcal{F}}$ (see [12, 15]).

In addition to our results on V_1, we study isometries between Lorentz sequence spaces. For a strictly decreasing weight sequence w, the Lorentz space $d(w, 1)$ is a classical sequence space with a unique symmetric Schauder basis (see [21, Section 4.e]). Using the description of the extreme points of its unit ball from [19, Theorem 2.6], we prove that two Lorentz spaces $d(w, 1)$ and $d(v, 1)$ are isometric if and only if the weight sequences are proportional (Theorem 5.5). This yields that the isometry groups of $d(w, 1)$ and of its predual $d_*(w, 1)$ are standard (Theorem 5.7), in agreement with results implicit in [10].

The paper is divided into two independent and self-contained parts. The first part (Sections 2–4) is devoted to the James-Schreier space V_1. We first introduce the space and establish structural properties of its dual, which allow us to characterize the extreme points of $B_{V_1^*}$ and deduce that V_1 has trivial isometry group. The second part (Section 5) contains a Banach-Stone-type theorem for Lorentz sequence spaces.

We use standard notation from Banach space theory. For a Banach space X, we denote by X^* its dual space, by B_X its closed unit ball, by S_X its unit sphere, and by $\operatorname{ext}(B_X)$ the set of extreme points of B_X, the points $x \in B_X$ such that whenever $x = \frac{y+z}{2}$ with $y, z \in B_X$, then $x = y = z$. If X has a Schauder basis (e_n) and $x = \sum_{n=1}^{\infty} x_n e_n$, we denote $\operatorname{supp} x = \{n \in \mathbb{N} : x_n \neq 0\}$. We also denote x_i by λ_i or $e_i^*(x)$. When $(e_n)_n$ is a shrinking basis of X, the sequence $(e_n^*)_n$ is a basis for X^* and we denote the coordinates of a vector $x^* \in X^*$ by x_i^* or $x^*(e_i)$. For a standard reference on Banach space theory, we refer the reader to [22]. $\mathbb{N}$ denotes

the set of strictly positive integers. We refer to the book [16] and the survey [1] for detailed presentations on the study of isometries over function spaces and over sequence spaces, respectively.

2. The James-Schreier space V_1 and its dual

The James-Schreier spaces V_p, for $p \geq 1$, were introduced and studied in [8, 7, 9]. In this work we will focus on the James-Schreier space V_1. Let us recall its definition.

Definition 2.1. We say that a subset $A \subseteq \mathbb{N}$ is permissible if $2 \leq |A| \leq \min A + 1$. Denote by $\mathbb{P}$ the family of all permissible sets and $\mathbb{P}^{\text{MAX}}$ the permissible sets that are maximal with respect to inclusion.

The definition of the Schreier family varies from reference to reference, but we can think of permissible sets as corresponding to those elements of the Schreier family which are not singletons. It will be convenient to enumerate any nonempty subset F of some permissible set A as $F = \{n_1 < n_2 < \ldots < n_{k+1}\}$, where $0 \leq k \leq n_1$. F is a singleton when $k = 0$; it is an element of $\mathbb{P}$ when $1 \leq k \leq n_1$; and F is an element of $\mathbb{P}^{\text{MAX}}$ when $k = n_1$.

Given a sequence $x = (x_n)$ of real numbers, denoting $\nu(x, A) \coloneqq \sum_{j=1}^{k} |x_{n_j} - x_{n_{j+1}}|$ for each $A \in \mathbb{P}$, we define:

$$\|x\|_{W_1} = \sup_{A \in \mathbb{P}} \nu(x, A) \leq \infty \quad \text{and} \quad W_1 \coloneqq \{x \in c_0 : \|x\|_{W_1} < \infty\}.$$

It follows that $(W_1, \|\cdot\|_{W_1})$ is a Banach space. Moreover, if (e_n) is the sequence of standard unit vectors in c_{00}, then each e_n clearly belongs to W_1, so that we can consider its closed subspace $V_1 \coloneqq \overline{\text{span}}\{e_n : n \in \mathbb{N}\} \subseteq W_1$, which is called the James-Schreier space (for $p = 1$).

From now on, we will omit the subscript W_1 in the norm of V_1. The following facts about the James-Schreier space hold: in [8] the authors prove that (e_n) is a Schauder basis of V_1 which is not unconditional and that V_1 is c_0-saturated. Moreover, it is proved in [9] that the basis (e_n) is shrinking, so that the sequence of biorthogonal functionals (e_n^*) is a Schauder basis of V_1^*.

Our main purpose is to describe the group of surjective linear isometries of V_1, $\text{Isom}(V_1)$, in terms of permutations of the basis (e_n) and possible changes of signs. To achieve our goal, we will characterize the extreme points of dual unit ball $B_{V_1^*}$ and describe $\text{Isom}(V_1^*)$.

Definition 2.2. We say that a functional $x^* \in B_{V_1^*}$ is compatible if there are $k \geq 0$ and $F = \{n_1 < \ldots < n_{k+1}\} \subseteq A \in \mathbb{P}$ such that:

$$x^* = \pm \left(\sum_{i=1}^{k+1} (-1)^i \lambda_i e_{n_i}^* \right) \text{ where } \begin{cases} \lambda_1 = 1, \\ \lambda_i = 2 \text{ if } 1 < i < k+1, \\ \lambda_{k+1} \in \{1, 2\} \text{ and } \lambda_{k+1} = 1 \text{ if } F \in \mathbb{P}^{\text{MAX}}. \end{cases}$$

Notice that if $x^* = \sum_{i=1}^{\infty} x_i^* e_i^*$ is a compatible functional, then any partial sum $\sum_{i=1}^{m} x_i^* e_i^*$ is also a compatible functional, if $m \geq \min \operatorname{supp} x^*$.

The following example will be helpful to estimate the norm of vectors:

Example 2.3. Given $A = \{n_1 < \ldots < n_{k+1}\} \in \mathbb{P}$ and a sequence $\vec{\theta} = (\theta_1, \ldots, \theta_k)$ of signs in $\mathbb{R}$, the functional $x^*_{A,\vec{\theta}} \coloneqq \sum_{i=1}^{k} \theta_i (e^*_{n_i} - e^*_{n_{i+1}}) \in V_1^*$ is compatible. Moroever, every compatible functional $x^* = \pm \left(\sum_{i=1}^{k+1} (-1)^i \lambda_i e^*_{n_i} \right)$ with $k \geq 1$ and $\lambda_{k+1} = 1$ is of the form $x^*_{A,\vec{\theta}}$ for some $A \in \mathbb{P}$ and some $\vec{\theta}$.

In fact, it is straightforward to check that $x^*_{A,\vec{\theta}} \in B_{V_1^*}$, furthermore if one writes:

$$x^*_{A,\vec{\theta}} = \theta_1 e^*_{n_1} + \underbrace{(\theta_2 - \theta_1)}_{\text{0, 2 or -2}} e^*_{n_2} + \cdots + \underbrace{(\theta_k - \theta_{k-1})}_{\text{0, 2 or -2}} e^*_{n_k} - \theta_k e^*_{n_{k+1}},$$

it is easy to verify that consecutive coordinates on $\operatorname{supp}\left(x^*_{A,\vec{\theta}}\right)$ have alternating signs, thus it is a compatible functional.

Proposition 2.4. *The set $M = \{x^* \in V_1^* : x^* \text{ is compatible}\} \cup \{0\}$ is a norming set that is symmetric and weak*-closed.*

Proof. Observe that M is a norming set because in Example 2.3 we proved that each $x^*_{A,\vec{\theta}}$ is compatible. Hence, given any $x = (x_n) \in V_1$ and $A = \{n_1 < \ldots < n_{k+1}\} \in \mathbb{P}$, let $\theta_i = \operatorname{sgn}(x_{n_i} - x_{n_{i+1}})$ and notice that $\nu(x, A) = x^*_{A,\vec{\theta}}(x)$. Moreover, it is clear that M is symmetric. We shall prove that M is weak*-closed.

Let $x^* \in \overline{M}^{w^*}$ with $x^* \neq 0$, we will prove that x^* is compatible. Because the canonical basis (e_n) of V_1 is shrinking (see [9]), we can write $x^* = \sum_{i=1}^{\infty} x^*(e_i) e_i^*$ with $x^*(e_i) \to 0$. It follows that:

$$|x^*(e_i)| \in \overline{\{|y^*(e_i)| : y^* \in M\}} = \{0, 1, 2\} \quad \text{for all } i \in \mathbb{N}.$$

Thus, $\operatorname{supp} x^*$ is finite. Let $S \coloneqq \operatorname{supp} x^* = \{n_1, \ldots, n_{k+1}\}$, with $k \geq 0$.

We consider the basic weak*-open neighbourhood of x^* given by:

$$U \coloneqq U\left(x^*, \{e_1, \ldots, e_{n_{k+1}}\}, \frac{1}{2}\right) = \left\{ y^* \in V_1^* : \max_{i \leq n_{k+1}} |(y^* - x^*)(e_i)| < \frac{1}{2} \right\}.$$

Let $y^* \in M \cap U$ and $i \leq n_{k+1}$. Since y^* is compatible, we obtain that

$$i \in S \implies |y^*(e_i) - x^*(e_i)| < \frac{1}{2} \implies y^*(e_i) = x^*(e_i)$$

and

$$i \notin S \implies |y^*(e_i)| = |y^*(e_i) - x^*(e_i)| < \frac{1}{2} \implies y^*(e_i) = x^*(e_i) = 0.$$

In particular, $S \subseteq \operatorname{supp} y^*$ and for every $i \in S$, $x^*(e_i) = y^*(e_i)$. Moreover, if $S \in \mathbb{P}^{\mathrm{MAX}}$, then $S = \operatorname{supp} y^*$ and, therefore, $|x^*(e_{n_{k+1}})| = 1$. It follows easily that, for $m = \max \operatorname{supp} y^*$, $x^* = \sum_{n=1}^{m} y^*_n e^*_n$, which is a compatible functional. □

We remember the following result, stated as Lemma 4 of [11]:

Lemma 2.5. *Let X be a Banach space over $\mathbb{K}$ and $N \subseteq B_{X^*}$ be a sign invariant norming set for X. Then $B_{X^*} = \overline{\operatorname{conv}(N)}^{w^*}$.*

Notice that for real Banach spaces, being sign invariant is the same as being symmetric. As a consequence of Proposition 2.4, Lemma 2.5 and Milman's Theorem (see [24, Theorem 3.25]), it follows that $\operatorname{ext}\left(B_{V_1^*}\right) \subseteq M$.

In what follows, our main goal is to determine which elements of M are extreme points of $B_{V_1^*}$. We present preliminary results which will be used in Section 3 to show that certain compatible functionals are indeed extreme points. Given a compatible functional $x^* = \sum_{i=1}^{k+1} x^*_{n_i} e^*_{n_i}$, we start with $y^*, z^* \in B_{V_1^*}$ such that $y^* + z^* = 2x^*$ and show that, under certain assumptions, $y^*_j = z^*_j = x^*_j$ for every $j \in \mathbb{N}$. Evaluating these functionals on specific vectors in the unit ball of X will be crucial for that purpose.

Notice that if $x \in V_1$, then $x \in c_0$, so that

$$|x(i)| = \lim_{j\to\infty} \nu(x, \{i, j\}) \leq \sup_{A\in\mathbb{P}} \nu(x, A) = \|x\|.$$

Therefore, $\|x\|_\infty \leq \|x\|$, thus $B_{V_1} \subseteq [-1, 1]^{\mathbb{N}}$.

Let us introduce some further notation. Given a set $A \subseteq \mathbb{N}$, let $\chi_A \coloneqq \sum_{j\in A} e_j$ and $x^*|_A \coloneqq x^*|_{\operatorname{span}\{e_i : i\in A\}}$, when $x^* \in V_1^*$. We also use interval notation within $\mathbb{N}$: $[i, j] = \{k \in \mathbb{N} : i \leq k \leq j\}$, $[i, j[= \{k \in \mathbb{N} : i \leq k < j\}$, $]i, j[= \{k \in \mathbb{N} : i < k < j\}$ and $]i, +\infty[= \{k \in \mathbb{N} : k > i\}$.

We start by pointing out a collection of vectors which are in B_{V_1}:

Lemma 2.6. *The following vectors are in the unit ball B_{V_1}:*

(1) e_i, *for* $i \leq 2$;
(2) $\frac{e_i}{2}$, *for* $i \geq 3$;
(3) $e_1 + \frac{e_j}{2}$, *for* $j \geq 2$;
(4) $\frac{e_i \pm e_j}{2}$, *for* $i \leq 2 \leq j$;
(5) $\chi_{[1,j]}$, *for* $j \geq 2$;
(6) $\frac{\chi_{[i,j]}}{2}$, *for* $2 \leq i < j$.

In what follows, we will use Lemma 2.6 several times without explicit mention. Any vector which is claimed to be in B_{V_1} should be listed above. The following lemmas, despite being simple, exemplify the use of these vectors to compute certain coordinates.

Lemma 2.7. *Given $y^*, z^* \in B_{V_1^*}$, the following assertions hold:*

(1) *If $i \leq 2$ and $y^*_i + z^*_i = \pm 2$, then $y^*_i = z^*_i = \pm 1$.*
(2) *If $i > 2$, $y^*_i + z^*_i = \pm 2$ and $y^*|_{[1,i[} = z^*|_{[1,i[} \equiv 0$, then $y^*_i = z^*_i = \pm 1$.*
(3) *If $i \leq 2$, $y^*_i + z^*_i = 0$ and $y^*_j = z^*_j = \pm 2$ for some $j > 2$, then $y^*_i = z^*_i = 0$.*

Proof. In (1) and (2), we get that $|y^*_i|, |z^*_i| \leq 1$ by evaluating y^* and z^* on the unit vectors e_i and $\chi_{[1,i]}$, respectively. It follows that $y^*_i = z^*_i = \pm 1$. To prove (3), we evaluate y^* and z^* on $\frac{e_i+e_j}{2} \in B_{V_1}$, obtaining either $\left|\frac{y^*_i}{2} + 1\right|, \left|\frac{z^*_i}{2} + 1\right| \leq 1$ or $\left|\frac{y^*_i}{2} - 1\right|, \left|\frac{z^*_i}{2} - 1\right| \leq 1$, and both alternatives imply that $y^*_i = z^*_i = 0$. □

Lemma 2.8. *Let $x^* = \sum_{i=1}^{k+1} x^*_{n_i} e^*_{n_i}$ be a compatible functional and $y^*, z^* \in B_{V_1^*}$ be such that $y^* + z^* = 2x^*$.*

(1) *If $|\operatorname{supp} x^*| > 2$, then $y^*|_{]n_1,n_{k+1}[} = z^*|_{]n_1,n_{k+1}[} = x^*|_{]n_1,n_{k+1}[}$.*
(2) *Moreover, if also $|x^*_{n_{k+1}}| = 2$, then $y^*|_{]n_1,+\infty[} = z^*|_{]n_1,+\infty[} = x^*|_{]n_1,+\infty[}$.*

Proof. We may assume without loss of generality that $x^*_{n_1} = 1$. Notice that $|\operatorname{supp} x^*| > 2$ implies that $n_1 \geq 2$. For $1 < i \leq k+1$, since $\frac{e_{n_i}}{2} \in B_{V_1}$, the evaluation of y^* and z^* yields $|y^*_{n_i}|, |z^*_{n_i}| \leq 2$. If $i < k+1$, since $y^*_{n_i} + z^*_{n_i} = 2x^*_{n_i} = \pm 4$, we get that $y^*_{n_i} = z^*_{n_i} = x^*_{n_i}$.

Let $1 < i < k+1$ and $n_i < j < n_{i+1}$ be such that for every $n_i < m < j$, $y^*_m = z^*_m = 0$. Since $\|\frac{\chi_{[n_i,j]}}{2}\| \leq 1$, we can evaluate y^* and z^* to get

$$\text{either} \quad \left|1 + \frac{y^*_j}{2}\right|, \left|1 + \frac{z^*_j}{2}\right| \leq 1 \quad \text{or} \quad \left|1 - \frac{y^*_j}{2}\right|, \left|1 - \frac{z^*_j}{2}\right| \leq 1.$$

Both imply that $y^*_j = z^*_j = 0$. If $n_1 < j < n_2$, a similar inductive argument applies to show that $y^*_j = z^*_j = 0$, by evaluating y^* and z^* on $\frac{\chi_{[j,n_2]}}{2} \in B_{V_1}$.

Finally, if $|x^*_{n_{k+1}}| = 2$, we use vectors $\frac{e_{n_{k+1}}}{2}$ and $\frac{\chi_{[n_{k+1},j]}}{2}$ to show that $y^*_{n_{k+1}} = z^*_{n_{k+1}} = x^*_{n_{k+1}}$ and $y^*_j = z^*_j = 0$ for $j > n_{k+1}$, which concludes the proof. □

For the remaining coordinates, it will be necessary to compute the norm of other vectors of V_1. To tackle the initial segment $[1, n_1[$ and the tail $]n_{k+1}, +\infty[$, we need the following remark:

Remark 2.9. Given a finitely supported vector $x \in V_1$, let $n = \min \operatorname{supp} x$. For $i \leq n-1$ and $t = x(n)$, we have

$$\|t\chi_{[i,n-1]} + x\| \leq \|x\|.$$

Similarly, if $n = \max \operatorname{supp} x$, for $j \geq n+1$ and $t = x(n)$, we have

$$\|x + t\chi_{[n+1,j]}\| \leq \|x\|.$$

The points defined on Lemma 2.10 will be used to determine the coordinates n_1 and n_{k+1}.

Lemma 2.10. *Given $F = \{n_1 < \ldots < n_m\}$ and $\vec{\theta} = (\theta_{n_i})_{i=1}^m \in \{-1, 1\}^F$, consider the following vector:*

$$y_{\vec{\theta}} \coloneqq \sum_{i=1}^{m} \theta_{n_i} e_{n_i}.$$

(1) *If $m = n_1$, then $\|y_{\vec{\theta}}\| \leq 2m - 1$.*
(2) *If $m = n_1 + 1$, then $\|y_{\vec{\theta}}\| \leq 2(m-1)$.*
(3) *If $m \geq n_1$ and $\theta_{n_1} = \theta_{n_2}$, then $\|y_{\vec{\theta}}\| \leq 2(m-1)$.*

Proof. For fixed F and $\vec{\theta}$, notice that the values of the coordinates of $y_{\vec{\theta}}$ have at most $m-1$ jumps of height 2, between two coordinates in the support, and two jumps of height 1, associated to the first and last nonzero coordinates.

If $m = n_1$, then a permissible set A containing F may contain only one further element, so that $\nu(y_{\vec{\theta}}, A) \leq 2(m-1) + 1 = 2m - 1$, while a permissible set which does not contain F may count the two jumps of height 1, but necessarily misses at least one jump of height 2, so that $\nu(y_{\vec{\theta}}, A) \leq 2(m-2) + 2 = 2(m-1)$. Hence, $\|y_{\vec{\theta}}\| \leq 2m - 1$.

If $m = n_1 + 1$, then a permissible set A containing F equals F, since F is maximal. Then it misses the first and last jumps and this yields $\nu(y_{\vec{\theta}}, A) \leq 2(m-1)$. A permissible set missing at least one element of F may count the two jumps of height 1, but necessarily misses at least one jump of height 2, so that $\nu(y_{\vec{\theta}}, A) \leq 2(m-2) + 2 = 2(m-1)$. Therefore, $\|y_{\vec{\theta}}\| \leq 2(m-1)$.

Finally, if $m \geq n_1$ and $\theta_{n_1} = \theta_{n_2}$, the coordinates of $y_{\vec{\theta}}$ have at most $m-2$ jumps of height 2 - from θ_{n_i} to $\theta_{n_{i+1}}$ for $2 \leq i \leq m$ - and 4 jumps of height 1: from 0 to θ_{n_1}, then back to 0 and back to $\theta_{n_2} = \theta_{n_1}$, and finally a last jump from θ_{n_m} to 0.

Thus, a permissible set A containing n_1 has at most $n_1 + 1 \leq m + 1$ elements. Hence, either A contains F and one more element, which makes it consider all steps of height 2 and at most 2 more jumps, which ensures $\nu(y_{\vec{\theta}}, A) \leq 2(m-2) + 2 = 2(m-1)$; otherwise it misses at least one jump of height 2 and may consider all 4 jumps of height 1, which yields $\nu(y_{\vec{\theta}}, A) \leq 2(m-3) + 4 = 2(m-1)$. Finally, A does not contain n_1, then it necessarily does not count two jumps of height 1, so that $\nu(y_{\vec{\theta}}, A) \leq 2(m-2) + 2 = 2(m-1)$. This concludes the proof. □

3. Extreme points of $B_{V_1{}^*}$

In this section we will determine what are the compatible functionals $x^* = \sum_{i=1}^{k+1} x^*_{n_i} e^*_{n_i} \in M$ that lie on $\operatorname{ext}\left(B_{V_1^*}\right)$. To ease the reading, the main theorem of this section, Theorem 3.4, is split on smaller propositions that will deal with different types of compatible functionals, highlighting where the points defined on the previous section are used.

The structure of the arguments to show that a given $x^* \in M$ is an extreme point of $B_{V_1^*}$ will follow the pattern: we suppose that $y^*, z^* \in B_{V_1^*}$ are such that $y^* + z^* = 2x^*$ and argue that $y^* = z^*$. Notice that x^* is an extreme point if, and only if, $-x^*$ is an extreme point. Starting from a compatible functional x^* which we want to decide if it is an extreme point of $B_{V_1{}^*}$ or not, this allows us to suppose, without loss of generality, that the first coordinate on the support of x^* is $x^*_{n_1} = 1$.

Proposition 3.1. $\pm e_i^* \in \operatorname{ext}\left(B_{V_1^*}\right) \iff i \in \{1, 2\}$.

Proof. Without loss of generality we consider only e_i^*.

($\Rightarrow$) For $i \geq 3$, fix $0 < \delta < 1$ and consider:

$$y^* \coloneqq \delta e^*_{i-1} + (1-\delta)e^*_i \qquad \text{and} \qquad z^* \coloneqq -\delta e^*_{i-1} + (1+\delta)e^*_i.$$

Because $y^* + z^* = 2e_i^*$ with $y^* \neq z^*$, to prove that e_i^* is not extreme, we need to show that $y^*, z^* \in B_{V_1^*}$. Indeed, if $x \in B_{V_1}$, then:

$$|y^*(x)| = |\delta x_{i-1} + (1-\delta)x_i| \leq \delta\|x\| + (1-\delta)\|x\| = \|x\| \leq 1$$

and

$$|z^*(x)| = |-\delta x_{i-1} + (1+\delta)x_i| < |x_i - x_{i-1}| + |x_i|$$
$$= \lim_{n\to\infty} |x_i - x_{i-1}| + |x_i - x_n| = \lim_{n\to\infty} \underbrace{\nu(x, \{i-1, i, n\})}_{\leq 1} \leq 1.$$

Thus, $e_i^* \notin \operatorname{ext}\left(B_{V_1^*}\right)$ for $i \geq 3$.

($\Leftarrow$) Conversely, for $i = 1$, let $y^*, z^* \in B_{V_1^*}$ be such that $y^* + z^* = 2e_1^*$. From (1) of Lemma 2.7 we conclude that $y_1^* = z_1^* = 1$. For each $j \in \mathbb{N}$, we evaluate y^* and z^* on $\chi_{[1,j]} \in B_{V_1}$ to obtain, by induction on $j \geq 2$:

$$|1 + y_j^*| = \left|y^*(\chi_{[1,j]})\right| \leq 1 \qquad \text{and} \qquad |1 + z_j^*| = \left|z^*(\chi_{[1,j]})\right| \leq 1.$$

It follows that $y_j^*, z_j^* \leq 0$ and since $y_j^* + z_j^* = 0$, we conclude that $y_j^* = z_j^* = 0$. Hence, $y^* = z^* = e_1^*$ and $e_1^* \in \operatorname{ext}\left(B_{V_1^*}\right)$. The proof for $i = 2$ follows analogously. □

As we mentioned in the previous section, we will use the points defined on Lemma 2.6, Remark 2.9 and Lemma 2.10 to determine whether or not x^* is an extreme point when its last coordinate is such that $|x_{n_{k+1}}| = 1$. We present:

Proposition 3.2. *Let $x^* = \sum_{i=1}^{k+1} x^*_{n_i} e^*_{n_i} \in M$ be such that $k \geq 1$ and $|x^*_{n_{k+1}}| = 1$. Then:*

$$x^* \in \operatorname{ext}(B_{V_1^*}) \iff \operatorname{supp} x^* \in \mathbb{P}^{\mathrm{MAX}}.$$

Proof. ($\Rightarrow$) Suppose that $\operatorname{supp} x^* \notin \mathbb{P}^{\mathrm{MAX}}$. Let $l > n_{k+1}$ and define:

$$y^* := x^* + (e^*_{n_{k+1}} - e^*_l) \qquad \text{and} \qquad z^* := x^* - (e^*_{n_{k+1}} - e^*_l).$$

Since $\{n_1, \ldots, n_{k+1}, l\} \in \mathbb{P}$, it follows from Example 2.3 that y^* and z^* are compatible funcionals. In particular, $y^*, z^* \in B_{V_1^*}$. Since $y^* + z^* = 2x^*$, we get that $x^* \notin \operatorname{ext}\left(B_{V_1^*}\right)$.

($\Leftarrow$) Suppose that $\operatorname{supp} x^* \in \mathbb{P}^{\mathrm{MAX}}$, let $y^*, z^* \in B_{V_1^*}$ such that $y^* + z^* = 2x^*$ and assume, without loss of generality, that $x^*_{n_1} = 1$. To prove that x^* is an extreme point, we will divide the argument in the following two cases:

Case 1: $|\operatorname{supp} x^*| = 2$.

In this case, $x^* = e^*_1 - e^*_j$ for some $j > 1$. Then it follows from (1) of Lemma 2.7 that $y^*_1 = z^*_1 = 1$. Also, since $\dfrac{e_1 \pm e_j}{2} \in B_{V_1}$, we have:

$$\left|\frac{1}{2} + \frac{y^*_j}{2}\right| = \left|y^*\left(\frac{e_1 + e_j}{2}\right)\right| \leq 1 \quad \text{and} \quad \left|\frac{1}{2} - \frac{y^*_j}{2}\right| = \left|y^*\left(\frac{e_1 - e_j}{2}\right)\right| \leq 1$$

and we conclude that $|y^*_j| \leq 1$. Similarly, $|z^*_j| \leq 1$. Since $y^*_j + z^*_j = 2x^*_j = -2$, we get that $y^*_j = z^*_j = -1 = x^*_j$. Finally, to show that $y^* = z^*$, let $i > 1$, $i \neq j$. By evaluating y^* and z^* on $e_1 + \dfrac{e_i}{2} \in B_{V_1}$, we conclude that $y^*_i, z^*_i \leq 0$. Since $y^*_i + z^*_i = 0$, we get that $y^*_i = z^*_i = 0$. Thus $y^* = z^* = x^*$ and x^* is extreme.

Case 2: $|\operatorname{supp} x^*| > 2$.

Write $x^* = \sum_{i=1}^{k+1} x^*_{n_i} e^*_{n_i}$ and observe that $y^*|_{]n_1, n_{k+1}[} = z^*|_{]n_1, n_{k+1}[} = x^*|_{]n_1, n_{k+1}[}$, by Lemma 2.8.

Claim 1: $y^*_{n_1} = z^*_{n_1} = x_{n_1} = 1$.

Proof of Claim 1. Consider the vectors

$$y = e_{n_1} + \sum_{i=2}^{k} (-1)^{i-1} e_{n_i} \quad \text{and} \quad \bar{y} = e_{n_1} - \sum_{i=2}^{k} (-1)^{i-1} e_{n_i}.$$

Since $\operatorname{supp} x^* = \{n_1, \ldots, n_{k+1}\} \in \mathbb{P}^{\mathrm{MAX}}$, we get that $k = n_1$. Hence, $m := |\operatorname{supp} y| = |\operatorname{supp} \bar{y}| = k = n_1$. It follows from (1) of Lemma 2.10 that y and $\bar{y}$ have norm at most $2k - 1$.

The evaluation of y^* on them yields:

$$y^*_{n_1} + 2(k-1) = (2k-1) - (1 - y^*_{n_1})$$

and

$$y^*_{n_1} - 2(k-1) = -(2k-1) + (1 + y^*_{n_1}),$$

respectively. Using that $y^* \in B_{V_1}$, we get that $1 + y^*_{n_1} \geq 0$ and $1 - y^*_{n_1} \geq 0$, which implies that $|y^*_{n_1}| \leq 1$. Similarly, $|z^*_{n_1}| \leq 1$. Since $y^*_{n_1} + z^*_{n_1} = 2x^*_{n_1} = 2$, we conclude that $y^*_{n_1} = z^*_{n_1} = 1 = x^*_{n_1}$. □

Claim 2: $y^*|_{[1,n_1[} = z^*|_{[1,n_1[} = x^*|_{[1,n_1[} \equiv 0$.

Proof of Claim 2. Given $1 \leq j < n_1$, suppose that for all $j < m < n_1$ we have $y^*_m = z^*_m = 0$. The evaluation of y^* and z^* on $\chi_{[j,n_1-1]} + y$, by Remark 2.9 and Lemma 2.10 gives us that:

$$y^*_j + 2k - 1 \leq \|y^*\|\|\chi_{[j,n_1-1]} + y\| \leq \|y\| \leq 2k - 1$$

and

$$z^*_j + 2k - 1 \leq \|z^*\|\|\chi_{[j,n_1-1]} + y\| \leq \|y\| \leq 2k - 1,$$

so that $y^*_j, z^*_j \leq 0$. Since $y^*_j + z^*_j = 2x^*_j = 0$, this implies that $y^*_j = z^*_j = 0$. □

Claim 3: $y^*_{n_{k+1}} = z^*_{n_{k+1}} = x_{n_{k+1}} = 1$.

Proof of Claim 3. For the coordinates above n_{k+1}, let us consider the vectors

$$z = \sum_{i=1}^{k} (-1)^{i-1} e_{n_i} + e_{n_{k+1}} \quad \text{and} \quad \bar{z} = \sum_{i=1}^{k} (-1)^{i-1} e_{n_i} - e_{n_{k+1}}.$$

Since $\operatorname{supp} x^* = \{n_1, \ldots, n_{k+1}\} \in \mathbb{P}^{\mathrm{MAX}}$, we get that $k = n_1$. It follows from (2) of Lemma 2.10 that z and $\bar{z}$ have norm at most $2k$.

The evaluation of y^* on these vectors yields:

$$2k - 1 + y^*_{n_{k+1}} = 2k - (1 - y^*_{n_{k+1}}) \text{ and } 2k - 1 - y^*_{n_{k+1}} = 2k - (1 + y^*_{n_{k+1}}),$$

respectively. Since y^* has norm 1, it follows that $|y^*_{n_{k+1}}| \leq 1$. Similarly, $|z^*_{n_{k+1}}| \leq 1$. Thus, $y^*_{n_{k+1}} = z^*_{n_{k+1}} = x^*_{n_{k+1}}$. □

Claim 4: $y^*|_{]n_{k+1},+\infty[} = z^*|_{]n_{k+1},+\infty[} = x^*|_{]n_{k+1},+\infty[} \equiv 0$.

Proof of Claim 4. Let $j > n_{k+1}$ be such that $y^*_m = z^*_m = 0$ for every $n_{k+1} < m < j$. Suppose $x^*_{n_{k+1}} = 1$. Notice that, by Remark 2.9 and (2) of Lemma 2.10,

$$\left\|z + \chi_{[n_{k+1}+1,j]}\right\| = \|z\| \leq 2k.$$

Also,

$$y^*\left(z + \chi_{[n_{k+1}+1,j]}\right) = (1 + 2(k-1) + 1) + y^*_j = 2k + y^*_j.$$

This determines the sign of y^*_j. Similarly, the same inequality holds for z^*_j, so that $\operatorname{sgn}(y^*_j) = \operatorname{sgn}(z^*_j)$ and, since $y^*_j + z^*_j = 2x^*_j = 0$, we get that $y^*_j = z^*_j = 0$. In case $x^*_{n_{k+1}} = -1$, we replace $z + \chi_{[n_{k+1}+1,j]}$ by $\bar{z} - \chi_{[n_{k+1}+1,j]}$ and the same argument guarantees that $y^*_j = z^*_j = 0$. □

Thus, $x^* = y^* = z^*$ and $x^* \in \operatorname{ext}\left(B_{V_1^*}\right)$. □

The next proposition deals with the case where x^* is a compatible functional and its last non-zero coordinate is ± 2. We will use different techniques to decide whether or not x^* is an extreme point, depending on the cardinality of $\operatorname{supp} x^*$.

Proposition 3.3. *Let $x^* = \sum_{i=1}^{k+1} x^*_{n_i} e^*_{n_i} \in M$ be such that $|x^*_{n_{k+1}}| = 2$. Then:*

$$x^* \in \operatorname{ext}\left(B_{V_1^*}\right) \iff n_1 - 1 \leq |\operatorname{supp} x^*| \leq n_1.$$

Proof. Notice that $|x^*_{n_{k+1}}| = 2$ implies $k \geq 1$. For both implications we can assume, without loss of generality, that $x^*_{n_1} = 1$.

($\Rightarrow$) Let $F := \{n_1 < n_2 < \ldots < n_{k+1}\}$. If $|F| < n_1 - 1$, then the set

$$\{n_1 - 1\} \cup F \in \mathbb{P} \setminus \mathbb{P}^{\mathrm{MAX}}.$$

Let $0 < \delta < 1$ and define:

$$y^* := \delta e^*_{n_1-1} + (1-\delta)e^*_{n_1} + \sum_{i=2}^{k+1} x^*_{n_i} e^*_{n_i} \quad \text{and} \quad z^* := -\delta e^*_{n_1-1} + (1+\delta)e^*_{n_1} + \sum_{i=2}^{k+1} x^*_{n_i} e^*_{n_i}.$$

Clearly, $y^* + z^* = 2x^*$. To conclude the proof we need to show that $y^*, z^* \in B_{V_1^*}$. Given $y \in B_{V_1}$, then:

$$\begin{aligned} |y^*(y)| &= |\delta y_{n_1-1} + (1-\delta)y_{n_1} - 2y_{n_2} + \cdots (-1)^k 2y_{n_{k+1}}| \\ &= |\delta(y_{n_1-1} - y_{n_1}) + (y_{n_1} - y_{n_2}) - (y_{n_2} - y_{n_3}) + \cdots \\ &\quad + (-1)^{k-1}(y_{n_k} - y_{n_{k+1}}) + (-1)^k y_{n_{k+1}}| \\ &\leq |y_{n_1-1} - y_{n_1}| + |y_{n_1} - y_{n_2}| + \cdots + |y_{n_k} - y_{n_{k+1}}| + |y_{n_{k+1}}|. \end{aligned}$$

Notice that, for all $j > n_{k+1}$, $B_j := \{n_1 - 1, n_1, \ldots, n_{k+1}, j\} \in \mathbb{P}$ and $\nu(y, B_j) \leq 1$. Thus:

$$|y_{n_1-1} - y_{n_1}| + |y_{n_1} - y_{n_2}| + \cdots + |y_{n_k} - y_{n_{k+1}}| + |y_{n_{k+1}}| = \lim_{j\to\infty} \nu(y, B_j) \leq 1$$

and we conclude that $y^* \in B_{V_1^*}$. In the same way, one can prove that $z^* \in B_{V_1^*}$. Therefore, $x^* \notin \operatorname{ext}\left(B_{V_1^*}\right)$.

($\Leftarrow$) Conversely, let $x^* = \sum_{i=1}^{k+1} x^*_{n_i} e^*_{n_i} \in M$ with $k \geq 1$, $x^*_{n_1} = 1$, $|x^*_{n_{k+1}}| = 2$ and let $y^*, z^* \in B_{V_1^*}$ be such that $y^* + z^* = 2x^*$. We split the proof in the following cases:

Case 1: $|\operatorname{supp} x^*| = n_1$.

It follows directly from Lemma 2.8 that $y^*|_{]n_1,+\infty[} = z^*|_{]n_1,+\infty[} = x^*|_{]n_1,+\infty[}$. The proofs of the next two claims are similar to those of Claims 1 and 2 in the previous proposition.

Claim 1: $y^*_{n_1} = z^*_{n_1} = x_{n_1} = 1$.

Proof of Claim 1. Consider the vectors

$$y = e_{n_1} + \sum_{i=2}^{k+1} (-1)^{i-1} e_{n_i} \quad \text{and} \quad \bar{y} = e_{n_1} - \sum_{i=2}^{k+1} (-1)^{i-1} e_{n_i}.$$

Since $|\operatorname{supp} y| = |\operatorname{supp} \bar{y}| = k+1 = |\operatorname{supp} x^*| = n_1$, it follows from (1) of Lemma 2.10, that $\|y\|, \|\bar{y}\| \leq 2(k+1) - 1 = 2k+1$. The evaluation of y^* and z^* on these vectors yields $|y^*_{n_1}| \leq 1$ and $|z^*_{n_1}| \leq 1$. Because $y^*_{n_1} + z^*_{n_1} = 2$, we conclude that $y^*_{n_1} = z^*_{n_1} = x^*_{n_1} = 1$. □

Claim 2: $y^*|_{[1,n_1[} = z^*|_{[1,n_1[} = x^*|_{[1,n_1[} \equiv 0$.

Proof of Claim 2. Assume that $1 \leq j < n_1$ is such that for every $j < m < n_1$, $y^*_m = z^*_m = 0$. By evaluating y^* and z^* on $\chi_{[j,n_1-1]} + \bar{y}$, using Remark 2.9 and Lemma 2.10, we conclude that $y^*_j = z^*_j = 0$. Hence, by induction, we get that $y^*|_{[1,n_1[} = z^*|_{[1,n_1[} = x^*|_{[1,n_1[} \equiv 0$. □

This ensures that $y^* = z^* = x^*$, so that x^* is an extreme point and Case 1 is finished.

Case 2: $|\operatorname{supp} x^*| = n_1 - 1$.

Since $n_1 \geq 3$, from (3) of Lemma 2.7 and Lemma 2.8, it remains to prove the following claim:

Claim 3: $y^*|_{[3,n_1]} = z^*|_{[3,n_1]} = x^*|_{[3,n_1]}$.

Proof of Claim 3. If $n_1 = 3$, (2) of Lemma 2.7 gives that $y^*_{n_1} = z^*_{n_1} = x^*_{n_1}$, which concludes the proof.

If $n_1 > 3$, for each $3 \leq l < n_1$, let

$$w_l = -e^*_l + \sum_{i=2}^{n_{k+1}} (-1)^{i-1} e^*_{n_i}.$$

Since $|\operatorname{supp} w_l| = k + 1 = n_1 - 1$ and $\min \operatorname{supp} w_l = l \leq n_1 - 1$, it follows from (3) of Lemma 2.10 that $\|w_l\| \leq 2k$. When we evaluate y^* and z^* on w_l, we get that

$$|-y^*_l + 2k| \leq 2k \quad \text{and} \quad |-z^*_l + 2k| \leq 2k.$$

Hence, since $y^*_l + z^*_l = 2x^*_l = 0$, it follows that $y^*_l = z^*_l = x^*_l = 0$. Now we can apply (2) of Lemma 2.7 and conclude that $y^*_{n_1} = z^*_{n_1} = x^*_{n_1}$. □

Therefore, $x^* \in \operatorname{ext}\left(B_{V_1^*}\right)$. □

The conjunction of all propositions in this section allows us to characterize the extreme points of $B_{V_1^*}$ as follows:

Theorem 3.4. *An element $x^* \in B_{V_1^*}$ is an extreme point if, and only if, $x^* = \sum_{i=1}^{k+1} x^*_{n_i} e^*_{n_i}$ is a compatible functional satisfying one of the following conditions:*

(1) $x^* = \pm e^*_i$, *for* $i \in \{1, 2\}$;
(2) $\operatorname{supp} x^* \in \mathbb{P}^{\mathrm{MAX}}$;
(3) $|x^*_{n_{k+1}}| = 2$ *and* $n_1 - 1 \leq |\operatorname{supp} x^*| \leq n_1$.

4. Isometries of V_1

In this section we will use the set of all compatible functionals M, defined on Proposition 2.4, and the characterization of the extreme points in $B_{V_1^*}$ given in Theorem 3.4 to describe $\operatorname{Isom}(V_1)$ in terms of signed permutations.

In [11, 12] it is proved that the only permutation that is allowed when one attempts to describe the group of isometries of the Schreier space is the identity, although there is no restriction over the sequence of signs. Our theorem shows that, in the case of surjective isometries on the James-Schereier space, we are even more restrict: the only surjective isometries in $\operatorname{Isom}(V_1)$ are $\pm Id$. There are two crucial differences between the arguments used in the aforementioned papers and the study we have done so far: the first one is the fact that $\operatorname{ext}(B_{V_1^*})$ contains several extreme points with maximal support consisting of two points, namely the points of the form $e^*_1 - e^*_l$, and the second one is that the elements of $\operatorname{ext}(B_{V_1^*})$ have alternating signs in their coordinates. The additional extreme points will play an important role to prove that $S(e^*_n) = \pm e^*_{\sigma(n)}$, while the alternating signs of elements in $\operatorname{ext}(B_{V_1^*})$ will restrict the signs that are allowed in the description of $\operatorname{Isom}(V_1)$.

Proposition 4.1. *Suppose that $S : V_1^* \to V_1^*$ is a weak*-weak* continuous operator that preserves extreme points. Then, for all $n \in \mathbb{N}$, $S(e_n^*)$ is a compatible functional.*

Proof. For every $n \in \mathbb{N}$, using Theorem 3.4, it is straightforward to check that $e_n^* \in w^*\text{-cl}\left(\operatorname{ext} B_{V_1^*}\right)$. Furthermore, because the set of compatible functionals is weak*-closed and contains the extreme points of $B_{V_1^*}$, we have:

$$\begin{aligned} S(e_n^*) \in S\Big[w^*\text{- cl}\left(\operatorname{ext}\left(B_{V_1^*}\right)\right)\Big] &\subseteq w^*\text{- cl}\left(S\left[\operatorname{ext}\left(B_{V_1^*}\right)\right]\right) \\ &\subseteq w^*\text{- cl}\left(\operatorname{ext}\left(B_{V_1^*}\right)\right) \subseteq w^*\text{- cl}\,(M) = M. \end{aligned}$$ □

Proposition 4.2. *Let $S \in \operatorname{Isom}(V_1^*)$ be a w*-homeomorphism. For every $x^* \in \operatorname{ext}(B_{V_1^*})$:*

$$\operatorname{supp} x^* \in \mathbb{P}^{\mathrm{MAX}} \iff \operatorname{supp} S(x^*) \in \mathbb{P}^{\mathrm{MAX}}.$$

Proof. Let $x^* = \sum_{i=1}^{k+1} \lambda_i e_{n_i}^* \in \operatorname{ext}(B_{V_1^*})$ such that $\operatorname{supp} x^* \in \mathbb{P}^{\mathrm{MAX}}$. Then, $y^* = S(x^*) = \sum_{i=1}^{k'} \delta_i e_{m_i}^* \in \operatorname{ext}(B_{V_1^*})$. Suppose, towards a contradiction, that $\operatorname{supp} y^* \in \mathbb{P} \setminus \mathbb{P}^{\mathrm{MAX}}$. Let

$$\delta_{k'+1} = \begin{cases} -\operatorname{sgn}\delta_{k'} & \text{if } y^* = \pm e_1 \text{ or } |\operatorname{supp} y^*| = m_1, \\ -2\operatorname{sgn}\delta_{k'} & \text{if } y^* = \pm e_2 \text{ or } |\operatorname{supp} y^*| = m_1 - 1. \end{cases}$$

Notice that, for every $l > \operatorname{supp} y^*$, $y^* + \delta_{k'+1} e_l^* \in \operatorname{ext}(B_{V_1^*})$, so that

$$x^* + \delta_{k'+1} S^{-1}(e_l^*) = S^{-1}(y^* + \delta_{k'+1} e_l^*) \in \operatorname{ext}(B_{V_1^*}).$$

Moreover, since $(e_l^*)_l$ converges weak* to 0, so does $(S^{-1}(e_l^*))_l$. Hence, there is an increasing sequence $(l_i)_i$ such that $(\operatorname{supp} S^{-1}(e_{l_i}^*))_i$ is pairwise disjoint and $\operatorname{supp} S^{-1}(e_{l_i}^*) > \operatorname{supp} x^*$ for every i. This implies that $\operatorname{supp} S^{-1}(y^* + \delta_{k'+1} e_{l_i}^*) = \operatorname{supp} x^* \cup \operatorname{supp} S^{-1}(e_{l_i}^*) \notin \mathbb{P}$, contradicting the fact that $S^{-1}(y^* + \delta_{k'+1} e_l^*) \in \operatorname{ext}(B_{V_1^*})$ for $l_i > \operatorname{supp} y^*$. Thus, $\operatorname{supp} S(x^*) \in \mathbb{P}^{\mathrm{MAX}}$. Observe that the reverse implication follows from interchanging the roles of S and S^{-1}. □

Lemma 4.3. *Let $y^* = \sum_{i=1}^{k} \lambda_i e_{n_i}^*$ be a compatible functional such that $|\operatorname{supp} y^*| = n_1 > 1$ and $|\lambda_k| = 2$. If z^* is a compatible functional such that $y^* + z^* \in \operatorname{ext}(B_{V_1^*})$ and $\operatorname{supp}(y^* + z^*) \in \mathbb{P}^{\mathrm{MAX}}$, then $z^* = \pm e_l^*$ for some $l > n_k$.*

Proof. First, notice that y^* and z^* being compatible, if $|(y^* + z^*)(e_i)| = 1$, then either $|y^*(e_i)| = 1$ and $|z^*(e_i)| \in \{0, 2\}$ or $|z^*(e_i)| = 1$ and $|y^*(e_i)| \in \{0, 2\}$. Remark also that if $y^* + z^* \in \operatorname{ext}(B_{V_1^*})$ and $\operatorname{supp}(y^* + z^*) \in \mathbb{P}^{\mathrm{MAX}}$, then it has exactly two coordinates with absolute value equal to 1, $\min\operatorname{supp}(y^*+z^*)$ and $\max\operatorname{supp}(y^*+z^*)$.

Let $z^* = \sum_{i=1}^{k'} \delta_i e_{m_i}^*$. Let us first notice that $|\delta_{k'}| = 1$. Indeed, if $|\delta_{k'}| = 2$, then $\max\operatorname{supp}(y^* + z^*) \in \{m_1, n_1\}$. If $\max\operatorname{supp}(y^* + z^*) = m_1$, then $n_1 < m_1 < m_{k'} = n_k$ and the coordinates of y^* and z^* above m_1 cancel each other, so that $|\operatorname{supp}(y^* + z^*)| \leq |\operatorname{supp} y^*|$. Since $\operatorname{supp} y^* \notin \mathbb{P}^{\mathrm{MAX}}$ and $\min\operatorname{supp}(y^* + z^*) = \min\operatorname{supp} y^* = n_1$, we get that $\operatorname{supp}(y^* + z^*) \notin \mathbb{P}^{\mathrm{MAX}}$. If $\max\operatorname{supp}(y^* + z^*) = n_1$, similarly we get that $\operatorname{supp}(y^*+z^*) \notin \mathbb{P}^{\mathrm{MAX}}$, both cases contradicting the maximality of $\operatorname{supp}(y^* + z^*)$. Therefore, $|\delta_{k'}| = 1$.

Also, if $k' \geq 2$, we would get either three coordinates with absolute value equal to 1, $\{n_1, n_k, m_1\}$, or two of them would be equal and the only remaining one would have absolute value equal to 1, both cases contradicting the fact that $y^* + z^*$ has exactly two coordinates with absolute value equal to 1. Therefore, $k' = 1$ and it is easy to see that $m_1 > n_k$ in this case, which concludes the proof. □

In the next result we will prove that $\mathrm{Isom}(V_1)$ is trivial and this is obtained through a characterization of some isometries in $\mathrm{Isom}(V_1^*)$, as one would expect. We would like to highlight how important Lemma 4.3 is to control the interaction between the supports of two compatible functionals of the form $S(e_1^*)$ and $S(e_l^*)$.

Theorem 4.4. *If $S \in \mathrm{Isom}(V_1^*)$ is a weak*-homeomorphism, then $S = \pm Id$. In particular, we have $\mathrm{Isom}(V_1) = \{Id, -Id\}$.*

Proof. Let $S(e_1^*) = \sum_{i=1}^{k} \lambda_i e_{n_i}^*$ and we prove the following claim:

Claim 1. $|\operatorname{supp}\big(S(e_1^*)\big)| = n_1$.

Proof of Claim 1. For every $l > 1$, we have $e_1^* - e_l^* \in \mathrm{ext}(B_{V^*})$ and $\{1, l\} \in \mathbb{P}^{\mathrm{MAX}}$, so that $S(e_1^* - e_l^*) \in \mathrm{ext}(B_{V^*})$ and $\operatorname{supp} S(e_1^* - e_l^*) \in \mathbb{P}^{\mathrm{MAX}}$, by Proposition 4.2. Since $(e_l^*)_l$ converges weak* to 0, so does $(S(e_l^*))_l$. Hence, there is an increasing sequence $(l_i)_i$ such that $(\operatorname{supp} S(e_{l_i}^*))_i$ is pairwise disjoint. Fix i such that $\operatorname{supp} S(e_{l_i}^*) > \operatorname{supp} S(e_1^*)$. Since $S(e_{l_i}^*)$ is a compatible functional, their first coordinate has absolute value equal to 1. Also, $\operatorname{supp} S(e_1^* - e_l^*) \in \mathbb{P}^{\mathrm{MAX}}$, so that $S(e_1^* - e_l^*)$ has only two coordinates with absolute value equal to 1 - the minimum and the maximum of its support. Hence, we conclude that $|\operatorname{supp} S(e_{l_i}^*)| = 1$, so that $|\operatorname{supp} S(e_1^*)| = n_1$. □

Let us now show that S is induced by a signed permutation. It suffices to prove the following claim:

Claim 2. $|\operatorname{supp} S(e_l^*)| = 1$ for every $l \in \mathbb{N}$.

Proof of Claim 2. Recall that $y^* = S(e_1^*) = \sum_{i=1}^{k} \lambda_i e_{n_i}^*$ is an extreme point and it follows from Claim 1 that $|\operatorname{supp} y^*| = n_1$. Suppose, towards a contradiction, that $n_1 > 1$, in particular $|\lambda_k| = 2$ because $\operatorname{supp} y^* \notin \mathbb{P}^{\mathrm{MAX}}$.

Given $l \geq 2$ and let $z_l^* = S(e_l^*)$. It follows from Proposition 4.1 that z_l^* is a compatible functional and observe that $y^* - z_l^* = S(e_1^* - e_l^*) \in \mathrm{ext}(B_{V^*})$ and $\operatorname{supp}(y^* - z_l^*) \in \mathbb{P}^{\mathrm{MAX}}$ by Proposition 4.2. Now, Lemma 4.3 implies that $z_l^* = \pm e_{m_l}^*$ for some $m_l > n_k$.

The reader may now notice that $\mathrm{ran}(S) \subseteq \overline{span}(\{e_{m_l}^* : l \geq 2\} \cup \{y^*\})$, which yields that $e_1^* \notin \mathrm{ran}(S)$, a contradiction. Therefore $|\operatorname{supp} y^*| = n_1 = 1$ and we can write $y^* = \delta_1 e_{n_1}^*$. For all $l \geq 2$, in order to $\operatorname{supp} S(e_1^* - e_l^*) \in \mathbb{P}^{\mathrm{MAX}}$, we must have $|\operatorname{supp} S(e_l^*)| = 1$ and this concludes Claim 2. □

Claim 2 implies that there is a bijection σ of $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $S(e_n^*) = \varepsilon_n e_{\sigma(n)}^*$, where $|\varepsilon_n| = 1$. We fix such σ for the rest of the proof.

Claim 3. For all $n \in \mathbb{N}$, $\varepsilon_n = \varepsilon_1$ and $\sigma(n) = n$.

Proof of Claim 3. Observe that the signs of the coordinates of $S(e_1^* - e_n^*) = \varepsilon_1 e_1^* - \varepsilon_n e_{\sigma(n)}^*$ are alternating, thus $\varepsilon_n = \varepsilon_1$. Moreover, for every $n \in \mathbb{N}$, let $n = m_1 < \cdots <$

m_{n+1} be such that $\sigma(m_1) < \cdots < \sigma(m_{n+1})$ and notice that there is $x^* \in \operatorname{ext}(B_{V_1^*})$ such that $\operatorname{supp} x^* = \{m_1, \ldots, m_{n+1}\} \in \mathbb{P}^{\mathrm{MAX}}$. Hence,

$$S(x^*) = \sum_{i=1}^{n+1} \lambda_i S(e^*_{m_i}) = \sum_{i=1}^{n+1} \lambda_i e^*_{\sigma(m_i)} \in \operatorname{ext}(B_{V_1^*}),$$

and $\{\sigma(m_1), \ldots, \sigma(m_{n+1})\} = \operatorname{supp} S(x^*) \in \mathbb{P}^{\mathrm{MAX}}$. We conclude that $\sigma(n) = \sigma(m_1) = \min \operatorname{supp} S(x^*) = |\operatorname{supp} S(x^*)| - 1 = n$. □

From what we have done, we conclude that $S = \pm Id$. Now, let $T \in \operatorname{Isom}(V_1)$, it follows that $T^* = \pm Id_{V_1^*}$ and we have:

$$e^*_m(T(e_n)) = (e^*_m \circ T)(e_n) = T^*(e^*_m)(e_n) = \pm e^*_m(e_n),$$

and we conclude that $T = \pm Id_{V_1}$. □

5. A rigidity result on Lorentz sequence spaces

Given a decreasing sequence $w = (w_n)_{n\geq 1}$ of positive real numbers such that $w \in c_0 \setminus \ell_1$, we define the Lorentz sequence space, denoted by $d(w,1)$, as follows:

$$d(w,1) = \{x = (x_n)_{n\geq 1} \in \mathbb{R}^{\mathbb{N}} : \|x\|_{w,1} < \infty\},$$

where $\|x\|_{w,1} = \sup\limits_{\pi \in S_\infty} \sum\limits_{i=1}^{\infty} |x_{\pi(n)}| w_n$ and S_∞ denotes the set of all permutations of natural numbers. In this case, w is called a sequence of weights, the space $d(w,1)$ endowed with $\|\cdot\|_{w,1}$ is a Banach sequence space and the canonical vectors $(e_n)_{n\geq 1}$ form a symmetric Schauder basis. We will omit the index in the norm to improve readability.

It is well known that $\|x\|$ is attainend by its decreasing rearrangement, i.e., the sequence $\widetilde{x} = (\widetilde{x}_n)_{n\geq 1}$ that is obtained from rearranging $(|x_n|)_{n\geq 1}$ as a decreasing sequence via a permutation. In what follows, we will use the following notation: $W(n) = \sum\limits_{i=1}^{n} w_i$. The reader may notice that because $w \notin \ell_1$ is decreasing, we cannot have $w_n = 0$, thus $(W(n))_{n\geq 1}$ is strictly increasing. This will be important in the following results. We also include the following characterization of the extreme points of $d(w,1)$:

Theorem 5.1 ([19, Theorem 2.6])**.** *An element $x \in S_{d(w,1)}$ is an extreme point of $B_{d(w,1)}$ if, and only if, there exists $n_0 \in \mathbb{N}$ such that:*

$$\widetilde{x}_n = \begin{cases} \dfrac{1}{W(n_0)} & , \text{ if } n \leq n_0, \\ 0 & , \text{ if } n > n_0. \end{cases}$$

and $w_1 > w_{n_0}$, when $n_0 > 1$.

Without passing through the decreasing rearrangement $\widetilde{x}$ of x, the previous result states that $x \in \operatorname{ext} B_{d(w,1)}$ if, and only if, $x = \frac{1}{W(n_0)} \varepsilon \chi_A$, where $|A| = n_0$, $\varepsilon \in \{-1,1\}^A$ and, when A is not a singleton, $w_{n_0} < w_1$. Although this notation is longer than the one used in Theorem 5.1, the computations we will carry out in this section are done using the support of the elements in $d(w,1)$, thus the need of introducing the signs of each non-null coordinate.

Extending the language of [1], given two Banach sequence spaces X and Y and $\lambda \in \mathbb{R}$, we say that $\mathrm{Isom}(X,Y)$ is λ-*standard* if it is composed by all $T : X \to Y$ given by

$$T(x_n) = (\lambda \varepsilon_n x_{\pi(n)}), \tag{5.1}$$

where $\varepsilon = (\varepsilon_n)_n \in \{-1,1\}^{\mathbb{N}}$ is a sequence of signs and $\pi \in S_\infty$ is a permutation. We say that an operator satisfying equation (5.1) is induced by the scalar λ, the permutation π and the sequence of signs ε. Notice that if $X = Y$ and $\lambda = 1$, this coincides with the definition of standard group of isometries.

In this section we aim to prove that $\mathrm{Isom}\big(d(v,1), d(w,1)\big)$ is either empty or λ-standard, depending on the weight sequences. As a consequence, we obtain a Banach-Stone like result for Lorentz sequences spaces that is similar to the one presented in [13] for the Lorentz function spaces $L_{w,p}$ and, the space $d(w,1)$ and it's predual $d_*(w,1)$ have standard group of isometries. In our arguments, we will make use of certain vectors which are supported in 2 or 3 elements of the canonical basis and we will need them to be extreme points of the unit ball. This is guaranteed for spaces whose sequence of weights w is strictly decreasing. Hence, from now on we shall work under this assumption. In particular, we get from the previous theorem that

$$ext(B_{d_{w,1}}) = \left\{ \frac{1}{W(n)} \varepsilon \chi_A : n \in \mathbb{N}, |A| = n \text{ and } \varepsilon \in \{-1,1\}^A \right\}.$$

Lemma 5.2. *If $T \in \mathrm{Isom}\big(d(v,1); d(w,1)\big)$, then, for all $r, s \in \mathbb{N}$,*

$$\operatorname{supp} T(e_r) = \operatorname{supp} T(e_s) \qquad or \qquad \operatorname{supp} T(e_r) \cap \operatorname{supp} T(e_s) = \emptyset.$$

Proof. For each $r \in \mathbb{N}$, by Theorem 5.1, there is a finite $A_r \subseteq \mathbb{N}$ and $\varepsilon^r \in \{-1,1\}^{A_r}$ such that $T(e_r) = \frac{v_1}{W(n_r)} \varepsilon^r \chi_{A_r}$. Let $n_r = |A_r|$.

Given distinct $r, s \in \mathbb{N}$, suppose that $A_r \cap A_s \neq \emptyset$ and fix $q \in A_r \cap A_s$. Notice that the point

$$z := \frac{\varepsilon^r(q) e_r + \varepsilon^s(q) e_s}{V(2)} \in \operatorname{ext} B_{d(v,1)},$$

so that

$$T(z) = \frac{1}{V(2)} \left(\varepsilon^r(q) \frac{v_1}{W(n_r)} \varepsilon^r \chi_{A_r} + \varepsilon^s(q) \frac{v_1}{W(n_s)} \varepsilon^s \chi_{A_s} \right) \in \operatorname{ext} B_{d(w,1)}.$$

In the view of Theorem 5.1, all the non-zero coordinates of $T(z)$ equal to $\frac{1}{W(n)}$ in absolute value, where $n = |\operatorname{supp} T(z)|$.

Now, if $A_r \neq A_s$ and, without loss of generality, there is $p \in A_r \setminus A_s$, then

$$|T(z)(p)| = \frac{v_1}{V(2)W(n_r)} < \frac{v_1}{V(2)} \left(\frac{1}{W(n_r)} + \frac{1}{W(n_s)} \right) = |T(z)(q)|,$$

a contradiction. Hence, if $A_r \cap A_s \neq \emptyset$, then $A_r = A_s$. □

Lemma 5.3. *If $T \in \mathrm{Isom}\big(d(v,1); d(w,1)\big)$, then $|\operatorname{supp} T(e_r)| = |\operatorname{supp} T(e_s)|$ for all $r, s \in \mathbb{N}$.*

Proof. For each $r \in \mathbb{N}$, by Theorem 5.1, there is a finite $A_r \subseteq \mathbb{N}$ and $\varepsilon^r \in \{-1,1\}^{A_r}$ such that $T(e_r) = \frac{v_1}{W(n_r)} \varepsilon^r \chi_{A_r}$. Let $n_r = |A_r|$.

Suppose, towards a contradiction, that for some distinct r and s, $n_r < n_s$. Then, by Lemma 5.2, $A_r \cap A_s = \emptyset$.

Fix $a > 0$ and let $0 < b < \dfrac{aW(n_r)}{W(n_s)} < a$. Notice that $\dfrac{b}{W(n_s)} < \dfrac{a}{W(n_r)}$ and $\dfrac{b}{W(n_r)} < \dfrac{a}{W(n_s)}$, thus, we can compute:

$$av_1+bv_2 = \|ae_r+be_s\|_{v,1} = \|aT(e_r)+bT(e_s)\|_{w,1} = av_1+\frac{bv_1(W(n_s+n_r)-W(n_r))}{W(n_s)}$$

and

$$av_1+bv_2 = \|ae_s+be_r\|_{v,1} = \|aT(e_s)+bT(e_r)\|_{w,1} = av_1+\frac{bv_1(W(n_r+n_s)-W(n_s))}{W(n_r)}.$$

Manipulating the former equations yields:

$$\Big(W(n_r+n_s)-W(n_s)-W(n_r)\Big)\cdot\Big(W(n_s)-W(n_r)\Big)=0.$$

However, $W(n_r) < W(n_s)$ because $n_r < n_s$ implies the following:

$$W(n_r+n_s) = w_1+\cdots+w_{n_s}+\underbrace{w_{n_s+1}+\cdots w_{n_s+n_r}}_{<W(n_r)} < W(n_s)+W(n_r),$$

a contradiction. □

Lemma 5.4. *If $T \in \operatorname{Isom}\big(d(v,1);d(w,1)\big)$, then:*

$$\operatorname{supp} T(e_r)\cap\operatorname{supp} T(e_s)=\emptyset$$

and, therefore, $|\operatorname{supp} T(e_r)| = 1$ for every $r \in \mathbb{N}$.

Proof. Suppose $r, s \in \mathbb{N}$ are distinct such that $\operatorname{supp} T(e_r) = \operatorname{supp} T(e_s)$ and let t be such that $\operatorname{supp} T(e_r)\cap\operatorname{supp} T(e_t) = \emptyset$. Let $n = |\operatorname{supp} T(e_r)| = |\operatorname{supp} T(e_t)|$. Hence, the nonzero coordinates of $T(e_r), T(e_s), T(e_t)$ have absolute value equal to $\frac{v_1}{W(n)}$.

Then

$$\frac{1}{V(3)}(e_r+e_s+e_t)\in \operatorname{ext} B_{d(v,1)}.$$

It follows that

$$\frac{1}{V(3)}\left(T(e_r)+T(e_s)+T(e_t)\right)\in \operatorname{ext} B_{d(w,1)},$$

which implies that all nonzero coordinates of $T(e_r)+T(e_s)+T(e_t)$ have the same absolute value.

On the other hand, it is easy to see that the nonzero coordinates of $T(e_r)+T(e_s)$ have absolute value equal to $\frac{2v_1}{W(n)}$, while the nonzero coordinates of $T(e_t)$ have absolute value equal to $\frac{v_1}{W(n)}$. Since $\operatorname{supp}(T(e_r)+T(e_s))$ is disjoint from $\operatorname{supp} T(e_t)$, we get coordinates of $T(e_r)+T(e_s)+T(e_t)$ with different absolute values, a contradiction. □

We finally use the previous lemmas to prove the main result of this section:

Theorem 5.5. *Let $v, w \in c_0 \setminus \ell_1$ be stritcly decreasing. If $\operatorname{Isom}\big(d(v,1);d(w,1)\big)$ is non-empty, then there is $\lambda > 0$ such that $v = \lambda w$ and $\operatorname{Isom}\big(d(v,1);d(w,1)\big)$ is λ-standard.*

Proof. Let $T \in \mathrm{Isom}\big(d(v,1); d(w,1)\big)$. By Lemma 5.4, there is $\pi \in S_\infty$ such that, for every $n \in \mathbb{N}$, $T(e_n) = \lambda_n e_{\pi(n)}$ and since T is an isometry:

$$v_1 = \|e_n\|_{v,1} = \|T(e_n)\|_{w,1} = \|\lambda_n e_{\pi(n)}\|_{w,1} = |\lambda_n| w_1,$$

hence, for all $n \in \mathbb{N}$, $|\lambda_n| = \dfrac{v_1}{w_1}$ and we conclude that $T(e_n) = \lambda \varepsilon_n e_{\pi(n)}$ with $(\varepsilon_n) \in \{-1,1\}^{\mathbb{N}}$ and $\lambda = \frac{v_1}{w_1}$.

Now, by induction on n we prove that $\frac{v_n}{w_n} = \lambda$. Assume this holds for $n \in \mathbb{N}$, so that $V(n) = \lambda W(n)$. Notice that $\| \sum_{i=1}^{n+1} e_i\|_{v,1} = V(n+1)$, while

$$\|T(\sum_{i=1}^{n+1} e_i)\|_{w,1} = \lambda \| \sum_{i=1}^{n+1} \varepsilon_i e_{\pi(i)}\|_{w,1} = \lambda W(n+1).$$

Hence, $V(n) + v_{n+1} = V(n+1) = \lambda W(n+1) = \lambda(W(n) + w_{n+1})$, which implies that $v_{n+1} = \lambda w_{n+1}$.

Finally, it is easy to see that in case $v = \lambda w$, every operator $T : d(v,1) \to d(w,1)$ defined by $T(e_n) = \lambda \varepsilon_n e_{\pi(n)}$ is an isometry. □

It is easy to check that the standard unitary vectors (e_n) form a boundedly complete Schauder basis for $d(w,1)$, thus it admits a predual. It is well known that the predual of the Lorentz sequence spaces is:

$$d_*(w,1) = \left\{ x = (x_n) \in \mathbb{R}^{\mathbb{N}} : \lim_n \frac{1}{W(n)} \sum_{i=1}^{n} \widetilde{x}_i = 0 \right\},$$

endowed with the norm $\|(x_n)\|_W = \sup_n \dfrac{\sum_{i=1}^n \widetilde{x}_i}{W(n)}$. The Lorentz sequence space is a particular case of the Köthe space μ_w, and in the section 9 of [17], the author gives a description of both the dual and the predual of $d(w,1)$ passing by the α-dual space $\mu_w^\times$ (defined by Köthe and Toeplitz in [20]), although no explicit isometry is given and we couldn't find any references in the literature that describes the above identification via an isometry. On the other hand, on recent papers the dual and the predual of different constructions of Lorentz spaces are presented in [18] and in [14] and we decided to include a proof here for the sake of completeness:

Proposition 5.6. *The operator $\Phi_w : d_*(w,1)^* \to d(w,1)$ given by $\Phi_w(f) = (f(e_n))_{n\geq 1}$ is an isometry.*

Proof. We will check that Φ_w is well defined. Fix $f \in d_*(w,1)^*$, denote $a_i = f(e_i)$ and for each $\pi \in S_\infty$ and $n \in \mathbb{N}$, consider $y(\pi,n) = \sum_{i=1}^{n} \mathrm{sgn}(a_{\pi(i)}) w_i e_{\pi(i)}$. Then $y(\pi,n) \in S_{d_*(w,1)}$ and:

$$\sum_{i=1}^{n} |a_{\pi(i)}| w_i = f(y(\pi,n)) \leq \|f\| \quad \text{for all } n \in \mathbb{N}.$$

Thus, Φ_w well defined and $\|\Phi_w(f)\| \leq \|f\|$. On the other hand, if $y = (y_n) \in d(w,1)$, we will check that $f : d_*(w,1) \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} x_n y_n$, for $x = (x_n) \in d_*(w,1)$, is a bounded functional.

If $x = (x_n) \in d_*(w,1)$, by the definition of the norm, $S(n) \coloneqq \sum_{i=1}^{n} \widetilde{x}_i \leq W(n)\|x\|_W$ for each $n \in \mathbb{N}$. Thus:

$$\begin{aligned}\sum_{i=1}^{n} \widetilde{y}_i\, \widetilde{x}_i &= \sum_{i=1}^{n-1} \Big[\Big(\widetilde{y}_i - \widetilde{y}_{i+1}\Big) S(i)\Big] + \widetilde{y}_n S(n) \\ &\leq \|x\|_W \sum_{i=1}^{n-1} \Big[\Big(\widetilde{y}_i - \widetilde{y}_{i+1}\Big) W(i)\Big] + \widetilde{y}_n W(n) \\ &= \|x\|_W \sum_{i=1}^{n} \widetilde{y}_i w_i \leq \|x\|_W \|y\|_{w,1}.\end{aligned}$$

We conclude that f is well defined and $\|f\| \leq \|y\|_{w,1}$. Therefore, Φ_w is a surjective isometry. □

The next result shows that the isometries between preduals of Lorentz spaces inherit the rigidity aspects of their dual spaces:

Theorem 5.7. *Let $v, w \in c_0 \setminus \ell_1$ be stritcly decreasing. If $\operatorname{Isom}\big(d_*(v,1); d_*(w,1)\big)$ is non-empty, then there is $\lambda > 0$ such that $v = \lambda w$ and $\operatorname{Isom}\big(d_*(v,1); d_*(w,1)\big)$ is λ-standard. In particular, $d_*(w,1)$ has standard group of isometries.*

Proof. Let $T \in \operatorname{Isom}\big(d_*(v,1); d_*(w,1)\big)$ and recall that the adjoint operator $T^* : d_*(w,1)^* \to d_*(w,1)^*$ is also an isometry. We define an isometry S on $d(w,1)$ via the following (commutative) diagram, where Φ_v and Φ_w are given by Proposition 5.6:

$$\begin{array}{ccc} d(v,1) & \xleftarrow{\quad S \quad} & d(w,1) \\ {\scriptstyle \Phi_v}\uparrow & & \uparrow{\scriptstyle \Phi_w} \\ d_*(v,1)^* & \xleftarrow[\quad T^* \quad]{} & d_*(w,1)^* \end{array}$$

Notice that it follows from the definiton of Φ that $\Phi^{-1}(e_n) = e_n^*$ for all $n \in \mathbb{N}$ in each column of the diagram. By Theorem 5.5, $S(z_n) = (\lambda \varepsilon_n z_{\pi(n)})_{n\geq 1}$, where $\pi \in S_\infty$, $\varepsilon \in \{-1,1\}^{\mathbb{N}}$ and $\lambda = \dfrac{w_1}{v_1}$. Denoting $y = T(x)$, because the above diagram is commutative, for all $k \in \mathbb{N}$:

$$\begin{aligned} y_k = \Phi_w^{-1}(e_k)(y) = \Phi_w^{-1}(e_k)(T(x)) &= \Big[T^* \circ \Phi_w^{-1}(e_k)\Big](x) \\ = \Big[\Phi_v^{-1} \circ S(e_k)\Big](x) &= \Phi_v^{-1}(\lambda \varepsilon_k e_{\pi(k)})(x) = \varepsilon_k x_{\pi(k)}. \end{aligned}$$

Thus, T is induced by λ and $\pi \in S_\infty$. In particular, when $v_1 = w_1$, we conclude that $v = w$ and that $d_*(w,1)$ has standard group of isometries. □

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGPT in order to improve language and readability. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

References

[1] L. Antunes and K. Beanland. Surjective isometries on Banach sequence spaces: a survey. *Concr. Oper.*, 9:19–40, 2022.

[2] L. Antunes, K. Beanland, and H. V. Chu. On the geometry of higher order schreier spaces. *Illinois Journal of Mathematics*, 65(1):47–69, 2021.

[3] S. A. Argyros and M. González. The extreme points of the unit ball of the James space J and its dual spaces. *Studia Math.*, 282(3):257–287, 2025.

[4] S. Banach. Théorie des opérations linéaires, Éditions Jacques Gabay, Sceaux, 1993. *Reprint of the 1932 original*, 1932.

[5] S. F. Bellenot. Banach spaces with trivial isometries. *Israel J. Math.*, 56(1):89–96, 1986.

[6] S. F. Bellenot. Isometries of James space. Banach space theory, Proc. Res. Workshop, Iowa City/Iowa 1987, Contemp. Math. 85, 1-18 (1989)., 1989.

[7] A. Bird. An amalgamation of the Banach spaces associated with James and Schreier, Part II: Banach-algebra structure. In *Banach algebras 2009*, volume 91 of *Banach Center Publ.*, pages 35–43. Polish Acad. Sci. Inst. Math., Warsaw, 2010.

[8] A. Bird and N. J. Laustsen. An amalgamation of the Banach spaces associated with James and Schreier, Part I: Banach-space structure. In *Banach algebras 2009*, volume 91 of *Banach Center Publ.*, pages 45–76. Polish Acad. Sci. Inst. Math., Warsaw, 2010.

[9] A. Bird, N. J. Laustsen, and A. Zsák. Some remarks on James-Schreier spaces. *J. Math. Anal. Appl.*, 371(2):609–613, 2010.

[10] M. Š. Braverman and E. M. Semenov. Isometries of symmetric spaces. *Dokl. Akad. Nauk SSSR*, 217:257–259, 1974.

[11] C. Brech, V. Ferenczi, and A. Tcaciuc. Isometries of combinatorial Banach spaces. *Proc. Am. Math. Soc.*, 148(11):4845–4854, 2020.

[12] C. Brech and C. Piña. Banach-Stone-like results for combinatorial Banach spaces. *Ann. Pure Appl. Logic*, 172(8):13, 2021. Id/No 102989.

[13] N. L. Carothers, R. Haydon, and P. Lin. On the isometries of the Lorentz function spaces. *Isr. J. Math.*, 84(1-2):265–287, 1993.

[14] M. Ciesielski and G. Lewicki. Sequence lorentz spaces and their geometric structure. *The Journal of Geometric Analysis*, 29:1929–1952, 2019.

[15] M. Fakhoury. Isometries of p-convexified combinatorial Banach spaces. *Studia Math.*, 275(3):197–234, 2024.

[16] R. J. Fleming and J. E. Jamison. *Isometries on Banach spaces: Function spaces*, volume 129 of *Chapman Hall/CRC Monogr. Surv. Pure Appl. Math.* Boca Raton, FL: Chapman and Hall/CRC, 2003.

[17] D. J. H. Garling. On symmetric sequence spaces. *Proceedings of the London mathematical society*, 3(1):85–106, 1966.

[18] A. Kaminska and H. J. Lee. M-ideal properties in marcinkiewicz spaces. *Comment. Math*, pages 123–144, 2004.

[19] A. Kamińska, H. J. Lee, and G. Lewicki. Extreme and smooth points in Lorentz and Marcinkiewicz spaces with applications to contractive projections. *Rocky Mountain J. Math.*, 39(5):1533–1572, 2009.

[20] G. Koethe and O. Toeplitz. Lineare räume mit unendlich vielen koordinaten und ringe unendlicher matrizen. *Journal für die reine und angewandte Mathematik*, 1934.

[21] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces. I*, volume Band 92 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin-New York, 1977. Sequence spaces.

[22] R. E. Megginson. *An introduction to Banach space theory*, volume 183 of *Grad. Texts Math.* New York, NY: Springer, 1998.

[23] A. Moltó, V. Montesinos, and S. Troyanski. On quasi-denting points, denting faces and the geometry of the unit ball of $d(w, 1)$. *Arch. Math. (Basel)*, 63(1):45–55, 1994.

[24] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[25] D. Tan. Some new properties and isometries on the unit spheres of generalized James spaces J_p. *J. Math. Anal. Appl.*, 393(2):457–469, 2012.

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