



# An example distinguishing two convex sequential properties

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Dedicated to István Juhász, on the occasion of his 80th birthday, celebrating a lifetime of dancing feet and topological mind

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## ABSTRACT

Corson and Efremov introduced convex notions of countable tightness and the Fréchet-Urysohn property in the context of Banach spaces. We present an old unpublished example which consistently distinguishes these properties. Together with a recent result from [12], it yields that it is independent from ZFC whether these properties are equivalent or not.

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## 1. Introduction

Recall the following sequential properties of a given topological space  $X$ :

- $X$  has countable tightness if every point in the closure of a set  $F \subseteq X$  is in the closure of a countable subset of  $F$ .
- $X$  is Fréchet-Urysohn if every point in the closure of a set  $F \subseteq X$  is the limit of a sequence in  $F$ .

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It is clear that every Fréchet-Urysohn space has countable tightness and results by Balogh [2] and Fedorchuk [8] established the independence of the converse implication for compact spaces.

The main purpose of this note is to present a consistent example of a Banach space which distinguishes convex counterparts introduced by Corson and Efremov in [5] and [6] of the aforementioned topological properties. In fact, it distinguishes between the property of Corson and an intermediate property recently introduced by Martínez-Cervantes in [11].

**Definition 1.** Let  $X$  be a Banach space. Let us consider the following properties:

- $X$  has the property (C) of Corson if every family of closed convex subsets of  $X$  whose intersection is empty has a countable subfamily with empty intersection.
- $X$  has the property (E') if every weak\* sequentially closed convex set  $C \subseteq X^*$  is weak\* closed.
- $X$  has the property (E) of Efremov if every point in a bounded weak\* closed convex set  $C \subseteq X^*$  is the weak\* limit of a sequence in  $C$ .

Notice that (E) is an immediate convex analogue of Fréchet-Urysohn in the context of dual spaces and Pol proved in [16] the following characterization of (C), turning it into a convex analogue of tightness: a Banach space has property (C) if and only if every point of a bounded weak\* closed convex set  $C \subseteq X^*$  is in the weak\* closure of a countable subset of  $C$ . (E) clearly implies (E'), which in turn implies (C), see [12, Lemma 2]. Plichko and Yost ([15], pg. 352) asked whether (C) implies (E) and the main result of this note gives a consistent negative answer to this question:

**Theorem 2.** *It is consistent with ZFC that there is a compact Hausdorff scattered space  $K$  such that:*

- (i) *every finite power  $K^n$  of  $K$  is hereditarily separable;*
- (ii)  *$C(K)$  does not have property (E').*

Compact spaces are an important source of counterexamples for questions about the topology and the structure of Banach spaces. Given a compact Hausdorff space  $K$ , let  $C(K)$  be the Banach space of continuous scalar-valued functions defined on  $K$ , with the supremum norm. It is well-known that  $K$  is scattered if and only if  $C(K)$  is an Asplund space (i.e. every separable subspace of  $C(K)$  has separable dual), see [13]. Moreover, if all finite powers of  $K$  are hereditarily separable, then  $C(K)$  is weakly hereditarily Lindelöf (see e.g. [9, Theorem 4.38]). Any weakly Lindelöf Banach space has property (C), since closed convex sets are weakly closed (see [7, Theorem 3.19]). Hence, we get the following corollary:

**Corollary 3.** *It is consistent with ZFC that there is an Asplund space with property (C) and which does not have property (E').*

Martínez-Cervantes and Poveda proved in [12] that, under the Proper Forcing Axiom, every Banach space which has property (C) also has property (E'), establishing the independence of this statement. Another unpublished example of a space that has (C) and does not have (E) has been constructed by Justin T. Moore as a modification of Ostaszewski's space from [14] assuming the principle  $\diamond$ . An example of a Banach space with property (E') which fails property (E) has been given under the Continuum Hypothesis in [1], but the question whether the implication fails in ZFC remains open. We refer to [12] for a complete account on these and related problems.

Our construction appears originally in the author's PhD thesis [3] and was never published. It is a simplification of the construction made in [4], inspired by [10] and [17] of a locally compact Fréchet-Urysohn space of weight  $\omega_2$ . The construction in [4] gives a consistent example of a compact space of weight  $\omega_2$  with

hereditarily separable finite powers, which yields a consistent example of an Asplund space of density  $\omega_2$  with interesting structural properties. In the case of the present work, we run a similar construction replacing  $\omega_2$  by  $\omega_1$  as the underlying set of the topological space  $K$ . This makes the arguments simpler and allows us to analyse the sequences in the space  $C(K)^*$  and prove the main result.

In the next section we introduce the partial order used to force the existence of the space  $K$ . We recall some of its properties and how they provide some of the desired properties of the space  $K$ . In Section 3, we show Theorem 11, which contains the relevant information about weak\* convergence of sequences in  $C(K)^*$  and implies that  $C(K)$  fails to have property (E'). It is a modification of Rabus's Lemma 5.4 [17], where the convergence of points in the space  $K$  is analysed.

## 2. Preliminary lemmas and the construction of $K$

Let us fix the following notation:

**Definition 4** (Juhász, Soukup [10]). Given finite nonempty sets of ordinals  $x$  and  $y$  such that  $\max x < \max y$ , we define

$$x * y = \begin{cases} x \setminus y & \text{if } \max x \in y, \\ x \cap y & \text{if } \max x \notin y. \end{cases}$$

The following definition is a simplification of [10, Definition 2.1] replacing  $\omega_2$  by  $\omega_1$ :

**Definition 5.** Let  $\mathbb{P}$  be the forcing formed by conditions  $p = (D_p, h_p, i_p)$  where:

1.  $D_p \in [\omega_1]^{<\omega}$ ;
2.  $h_p : D_p \rightarrow \wp(D_p)$  and for all  $\xi \in D_p$ ,  $\max h_p(\xi) = \xi$ ;
3.  $i_p : [D_p]^2 \rightarrow [D_p]^{<\omega}$  and for all  $\xi, \eta \in D_p$ ,  $\xi < \eta$ , we have that:

- (a)  $h_p(\xi) * h_p(\eta) \subseteq \bigcup_{\gamma \in i_p(\{\xi, \eta\})} h_p(\gamma)$ ,
- (b)  $i_p(\{\xi, \eta\}) \subseteq \xi$ ;

ordered by  $p \leq q$  if  $D_p \supseteq D_q$ , for all  $\xi \in D_q$ ,  $h_p(\xi) \cap D_q = h_q(\xi)$  and  $i_p|_{[D_q]^2} = i_q$ .

The underlying set in [10, Definition 2.1] is  $\omega_2$  and the partial order depends on a function  $f : [\omega_2]^2 \rightarrow [\omega_2]^{\leq \omega}$  with the so called strong property  $\Delta$ . In our case, if we take  $f : [\omega_1]^2 \rightarrow [\omega_1]^{\leq \omega}$  to be defined by  $f(\{\xi, \eta\}) = \min\{\xi, \eta\}$ , then we have an exact analogue of Definition 2.1 in [10], where  $\omega_2$  is replaced by  $\omega_1$ . The role of the function  $f$  is to put some control on the image of the functions  $i_p$  in order to prove, for instance, that  $\mathbb{P}$  has the countable chain condition (ccc). Since in the case of the present work the underlying set is  $\omega_1$ , condition 3.(b) already limits the image of a pair  $\{\xi, \eta\}$  to a countable set. In particular, the following lemma follows from similar arguments as in [17, 10]:

**Lemma 6** (Rabus [17], Lemma 4.1; Juhász, Soukup [10], Lemma 2.8).  $\mathbb{P}$  satisfies ccc.

We will also need the following technical lemmas, whose versions in [10] consider the forcing with  $\omega_2$  as the underlying set, but still hold in our case:

**Lemma 7** (Lemma 2.2, [10]). For each  $\alpha < \omega_1$ , the set  $D = \{p \in \mathbb{P} : \alpha \in D_p\}$  is dense in  $\mathbb{P}$ .

**Lemma 8** (Lemma 2.16, [10]; [17]). Let  $t = (D_t, h_t, i_t) \in \mathbb{P}$ ,  $D_t = T \cup E \cup F$ , where  $T < E < F$ ,  $E = \{\alpha_1 < \dots < \alpha_k\}$ ,  $F = \{\alpha_i^1, \alpha_i^2 : 1 \leq i \leq k\}$ ,  $H \subseteq T$  and

$$\forall 1 \leq i \leq k \quad h_t(\alpha_i^1) \cap h_t(\alpha_i^2) = \bigcup_{\xi \in H \cup E} h_t(\xi).$$

Then there is  $u = (D_u, h_u, i_u) \in \mathbb{P}$  such that  $D_u = T \cup E$  and:

- (a)  $u \leq t|_T$ ;
- (b)  $u \leq t|_{H \cup E}$ ;
- (c)  $T \setminus \bigcup_{\xi \in H \cup E} h_t(\xi) \subseteq h_u(\alpha_1)$ .

Let us finally define the space  $K$ . Fix the ground model  $V$  and a generic filter  $G$ .

**Definition 9** (Juhász, Soukup [10], Definition 2.3). For each  $\xi < \eta < \omega_1$ , working in  $V[G]$ , let

$$h(\xi) = \bigcup_{p \in G} h_p(\xi) \quad \text{and} \quad i(\{\xi, \eta\}) = \bigcup_{p \in G} i_p(\{\xi, \eta\}),$$

and let  $L$  be the topological space  $(\omega_1, \tau)$ , where  $\tau$  is the topology on  $\omega_1$  which has the family of sets

$$\{h(\xi) : \xi < \omega_1\} \cup \{\omega_1 \setminus h(\xi) : \xi < \omega_1\}$$

as a topological subbasis.

It follows from [10, Theorem 1.5] that for all  $\xi < \omega_1$ ,  $h(\xi)$  is a compact subset of  $L$  and it is easy to check that

$$\{h(\xi) \setminus \bigcup_{\eta \in F} h(\eta) : F \in [\xi]^{<\omega}\}$$

forms a local topological basis at  $\xi$ . Therefore  $L$  is a locally compact scattered zero-dimensional space. In  $V^{\mathbb{P}}$ , let  $K$  be the one-point compactification of  $L$  and let us denote the point of compactification by  $\omega_1$ , i.e.  $K \setminus L = \{\omega_1\}$ . Then, we get the following result:

**Theorem 10** (Theorem 3.2, [4]). In  $V[G]$ ,  $K$  is a compact scattered zero-dimensional space such that  $K^n$  is hereditarily separable for every  $n \in \mathbb{N}$ .

The proof that  $K^n$  is hereditarily separable for every  $n \in \mathbb{N}$  in the forcing extension is done similarly to [4, Theorem 3.2]. Again there is a use of the function  $f$  with strong property  $\Delta$ , which guarantees that any uncountable family of conditions has a pair of conditions having three properties (i), (ii) and (iii) (see page 510 of [4]), which can therefore be amalgamated using [4, Lemma 2.7]. The existence of such a pair in our case follows with no use of the function  $f$ , since conditions (i), (ii) and (iii) get trivial when  $f(\{\xi, \eta\}) = \min\{\xi, \eta\}$ .

Theorem 10 guarantees the first part of Theorem 2. The next section is devoted to prove that the space  $K$  constructed in this section also satisfies assertion (ii) of Theorem 2, that is,  $C(K)$  fails to have property (E').

### 3. $C(K)$ does not have property (E')

Let us now prove the main result about the weak\* convergence of sequences in  $C(K)^*$ . Recall that by the Riesz Representation Theorem,  $C(K)^*$  can be seen as the space of bounded regular Borel measures on  $K$ . Given  $x \in K$ , we denote by  $\delta_x$  the point-evaluation functional, i.e.  $\delta_x(f) = f(x)$ .

It is well-known that bounded regular Borel measures on a compact scattered space  $K$  are atomic, see e.g. [18, Theorem 19.7.6]. This means that each  $\mu \in C(K)^*$  is of the form  $\sum_{x \in S} a_x \delta_x$  for some countable  $S \subseteq K$  and a sequence of nonzero scalars  $(a_x)_{x \in S}$  such that the series  $\sum_{x \in S} a_x$  converges absolutely. The variation of a measure  $\mu = \sum_{x \in S} a_x \delta_x \in C(K)^*$  on some set  $X \subseteq K$  is defined by  $|\mu|(X) = \sum_{x \in S \cap X} |a_x|$ . This characterization of the elements  $C(K)^*$  will be helpful in our proof.

**Theorem 11.** *In  $V[G]$ , if  $(\mu_n)_{n \in \mathbb{N}} \subseteq B_{C(K)^*}$  is a sequence weak\* convergent to  $\delta_{\omega_1}$ , then there is  $n_0 \in \mathbb{N}$  such that, for all  $n \geq n_0$ , we have that  $\mu_n(\{\omega_1\}) \neq 0$ .*

**Proof.** Suppose by contradiction that there is, in  $V[G]$ , a sequence  $(\mu_n)_{n \in \mathbb{N}} \subseteq B_{C(K)^*}$  that converges weakly\* to  $\delta_{\omega_1}$  and such that for all  $n \in \mathbb{N}$ ,  $\mu_n(\{\omega_1\}) = 0$ .

In  $V$ , let  $0 < \varepsilon < \frac{1}{5}$  and let  $\dot{\delta}_{\omega_1}$  be a  $\mathbb{P}$ -name for  $\delta_{\omega_1}$  and  $(\dot{\mu}_n)_{n \in \mathbb{N}}$  a sequence of names for elements of  $B_{C(K)^*}$  such that

$$\mathbb{P} \Vdash \forall n \in \mathbb{N} \ \dot{\mu}_n(\{\omega_1\}) = 0 \text{ and } (\dot{\mu}_n)_{n \in \mathbb{N}} \text{ converges weakly* to } \dot{\delta}_{\omega_1}.$$

Since  $K$  is scattered, it follows that each  $\mu_n$  is atomic. Therefore:

$$\mathbb{P} \Vdash \forall n \in \mathbb{N} \ \exists F_n \subseteq L \text{ finite such that } |\dot{\mu}_n|(K \setminus F_n) < \check{\varepsilon}.$$

For each  $n \in \mathbb{N}$ , let  $A_n$  be a maximal antichain in  $\mathbb{P}$  such that for every  $p \in A_n$  decides  $F_n$ , i.e. there exists a finite subset  $F_n^p$  of  $\omega_1$  such that  $p$  forces  $|\dot{\mu}_n|(K \setminus \check{F}_n^p) < \check{\varepsilon}$  and for every  $\alpha \in F_n^p$ , there is  $a_\alpha \in \mathbb{R}$  such that  $p$  forces that  $\dot{\mu}_n(\{\check{\alpha}\}) = \check{a}_\alpha$ . By Lemma 7, we can assume, without loss of generality, that for every  $n \in \mathbb{N}$  and every  $p \in A_n$ ,  $F_n^p \subseteq D_p$ .

From the fact that  $\mathbb{P}$  is ccc, it follows that there exists  $\gamma < \omega_1$  such that

$$\bigcup \{D_p : p \in A_n, n \in \mathbb{N}\} \subseteq \gamma.$$

Given  $q \in \mathbb{P}$ , since  $h_q(\gamma) \subseteq \gamma \subseteq \omega_1$  and, therefore,  $q \Vdash \omega_1 \notin h(\gamma)$ , it follows that  $q \Vdash \dot{\delta}_{\omega_1}(h(\gamma)) = 0$ . Since  $\mathbb{P}$  forces  $(\dot{\mu}_n)_{n \in \mathbb{N}}$  to converge weakly\* to  $\dot{\delta}_{\omega_1}$ , there are  $r \leq q$  and  $m \in \mathbb{N}$  such that

$$r \Vdash \forall n \geq m \ |\dot{\mu}_n(h(\gamma))| < \check{\varepsilon}.$$

Once again by Lemma 7, we can assume, without loss of generality, that  $\gamma \in D_r$ . Let  $H = D_r \cap \gamma$  and  $E = D_r \setminus \gamma = \{\gamma = \alpha_1 < \dots < \alpha_k\}$ . Let  $F \subseteq \omega_1$  be such that  $E < F$  and  $|F| = 2|E|$  and denote  $F = \{\alpha_i^1, \alpha_i^2 : 1 \leq i \leq k\}$ .

We will obtain, after 3 steps,  $u \in \mathbb{P}$  and  $n \in \mathbb{N}$  such that  $u \leq r$ ,  $n \geq m$  and  $u \Vdash |\dot{\mu}_n(h(\gamma))| > \check{\varepsilon}$ , contradicting the fact that  $r \Vdash \forall n \geq m \ |\dot{\mu}_n(h(\gamma))| < \check{\varepsilon}$ . In Step 1, we extend  $r$  to a condition  $s$  such that  $D_s = D_r \cup F$  and for every  $1 \leq i \leq k$ , we have  $h_s(\alpha_i^1) \cap h_s(\alpha_i^2) = \bigcup_{\xi \in D_r} h_s(\xi)$ ; in Step 2, we extend  $s$  to a condition  $t$  such that  $D_t \subseteq \gamma \cup E \cup F$  and for which there exist  $n \geq m$  and  $p \in A_n$  such that  $t \leq p$  and

$$t \Vdash |\dot{\mu}_n(\bigcup_{\xi \in D_r} h(\xi))| < \check{\varepsilon};$$

finally, in Step 3 we will obtain  $u$  such that  $D_u = (D_t \cap \gamma) \cup E$ ,  $u \leq r$  and

$$u \Vdash |\dot{\mu}_n(h(\gamma))| > \check{\varepsilon},$$

as desired.

**Step 1.** Define  $s = (D_s, h_s, i_s)$  by  $D_s = D_r \cup F$ ;

$$h_s(\xi) = \begin{cases} h_r(\xi) & \text{if } \xi \in D_r, \\ D_r \cup \{\xi\} & \text{if } \xi \in F; \end{cases}$$

and

$$i_s(\{\xi, \eta\}) = \begin{cases} i_r(\{\xi, \eta\}) & \text{if } \xi, \eta \in D_r, \\ \min\{\xi, \eta\} \cap D_s & \text{otherwise.} \end{cases}$$

Clearly  $s$  satisfies conditions 1 and 2 of Definition 5 and condition 3.(a) for  $\xi, \eta \in D_r$  follow from the fact that  $r \in \mathbb{P}$ .

Let  $\xi \in D_r$  and  $\eta \in F$ . Hence,  $\xi \in h_s(\eta)$  and  $h_s(\xi) \subseteq h_s(\eta)$  so that  $h_s(\xi) * h_s(\eta) = h_s(\xi) \setminus h_s(\eta) = \emptyset$ . Therefore,  $s$  satisfies condition 3.(a) of Definition 5 for these pairs.

Let now be  $\xi, \eta \in F$  and  $\xi < \eta$ . In this case,  $\xi \notin h_s(\eta)$  and hence,  $h_s(\xi) * h_s(\eta) = h_s(\xi) \cap h_s(\eta) = D_r$ . But  $D_r = H \cup E \subseteq \xi \cap D_s$ , so that  $s$  satisfies condition 3.(a) for these pairs.

We get that  $s \in \mathbb{P}$  and it is easy to see that  $s \leq r$ .

**Step 2.** Note that  $\mathbb{P}$  forces that  $\bigcup_{\xi \in \check{D}_r} h(\xi)$  is a clopen set and  $\omega_1 \notin \bigcup_{\xi \in \check{D}_r} h(\xi)$ . Hence,  $\mathbb{P}$  forces that  $\dot{\delta}_{\omega_1}(K \setminus \bigcup_{\xi \in \check{D}_r} h(\xi)) = 1$ .

Since  $\mathbb{P}$  forces that  $(\dot{\mu}_n)_{n \in \mathbb{N}}$  converges weakly\* to  $\dot{\delta}_{\omega_1}$ , there are  $t \leq s$  and  $n \geq m$  such that

$$t \Vdash \dot{\mu}_n(K \setminus \bigcup_{\xi \in \check{D}_r} h(\xi)) > 1 - \check{\varepsilon}.$$

But  $A_n$  is a maximal antichain and hence, we can assume, without loss of generality, that there exists  $p \in A_n$  such that  $t \leq p$ . Since  $t \leq p, r$ , we have that

$$t \Vdash \dot{\mu}_n(\check{F}_n^p \setminus \bigcup_{\xi \in \check{D}_r} h(\xi)) \geq \dot{\mu}_n(K \setminus \bigcup_{\xi \in \check{D}_r} h(\xi)) - |\dot{\mu}_n|(K \setminus \check{F}_n^p) > 1 - 2\check{\varepsilon},$$

i.e.,

$$\sum \{a_\alpha : \alpha \in F_n^p \setminus \bigcup_{\xi \in D_r} h_t(\xi)\} > 1 - 2\varepsilon.$$

**Step 3.** Let  $T = D_t \cap \gamma$  and observe that  $t, T, E, F$  and  $H$  satisfy the assumptions of Lemma 8. Hence, there exists  $u = (D_u, h_u, i_u) \in \mathbb{P}$  such that  $D_u = T \cup E$ ,  $u \leq t|_T$ ,  $u \leq t|_{H \cup E}$  and  $T \setminus \bigcup_{\xi \in H \cup E} h_t(\xi) \subseteq h_u(\alpha_1)$  and notice that  $t|_T \leq p$ ,  $H \cup E = D_r$  and  $t|_{H \cup E} = r$ .

It remains to show the statement below and we have a contradiction with the fact that  $u \leq r$  and that  $r \Vdash |\dot{\mu}_n(h(\gamma))| < \check{\varepsilon}$ :

**Claim.**  $u \Vdash \dot{\mu}_n(h(\gamma)) > \check{\varepsilon}$ .

**Proof of the claim.** Consider

$$I = \{\alpha \in F_n^p : t \Vdash \check{\alpha} \notin \bigcup_{\xi \in \check{D}_r} h(\xi)\} = F_n^p \setminus \bigcup_{\xi \in D_r} h_t(\xi)$$

and note that, since  $D_r = H \cup E$ ,  $\alpha_1 = \gamma$  and  $F_n^p \subseteq D_p \subseteq D_t \cap \gamma = T$ , we have that

$$I \subseteq T \setminus \bigcup_{\xi \in D_r} h_t(\xi) \subseteq h_u(\gamma).$$

As  $u \leq t|_T \leq p$  and  $p$  forces that  $\dot{\mu}_n(\{\check{\alpha}\}) = \check{a}_{\check{\alpha}}$  for every  $\alpha \in F_n^p$ , we have that

$$u \Vdash \dot{\mu}_n(\check{I}) = \sum_{\alpha \in \check{I}} \check{a}_{\alpha} > 1 - 2\check{\varepsilon},$$

and, since  $\mathbb{P}$  forces  $\|\dot{\mu}_n\| \leq 1$ , we have that

$$u \Vdash |\dot{\mu}_n|(h(\gamma) \setminus \check{I}) \leq |\dot{\mu}_n|(K \setminus \check{I}) \leq \|\dot{\mu}_n\| - |\dot{\mu}_n|(\check{I}) < 1 - (1 - 2\check{\varepsilon}) = 2\check{\varepsilon}.$$

Therefore,

$$u \Vdash \dot{\mu}_n(h(\gamma)) \geq \dot{\mu}_n(\check{I}) - |\dot{\mu}_n|(h(\gamma) \setminus \check{I}) > 1 - 4\check{\varepsilon} > \check{\varepsilon},$$

completing the proof of the claim and the theorem.  $\square$

In particular, it follows from the previous result that there is no sequence of points from  $L$  converging to  $\omega_1$  in  $K$ .

**Corollary 12.** *In  $V[G]$ ,  $C = \{\mu \in C(K)^* : \mu(\{\omega_1\}) = 0\}$  is a weak\* sequentially closed convex subset of  $C(K)^*$  which is not weak\* closed. Therefore,  $C(K)$  does not have property (E').*

**Proof.**  $C$  is clearly convex and it follows from Theorem 11 that  $C$  is weak\* sequentially closed. Indeed, since  $K$  is scattered, given a sequence  $(\mu_n)_{n \in \mathbb{N}}$  in  $C$ , it follows that each  $\mu_n$  is atomic, that is  $\mu_n = \sum_{k=1}^{\infty} a_k^n \delta_{\alpha_k^n}$  for some sequence  $(a_k^n)_{k \in \mathbb{N}}$  of scalars and some sequence  $(\alpha_k^n)_{k \in \mathbb{N}}$  of distinct elements of  $K$ . Since  $\mu_n \in C$ , we get moreover that  $(\alpha_k^n)_{k \in \mathbb{N}}$  is indeed a sequence in  $K \setminus \{\omega_1\} = L$ .

Given  $\mu \in C(K)^*$ , we can write  $\mu = \sum_{\alpha \in S} a_{\alpha} \delta_{\alpha}$  for some countable  $S \subseteq K$  and a sequence of scalars  $(a_{\alpha})_{\alpha \in S}$ . If  $\mu \notin C$ , then  $\omega_1 \in S$  and  $a_{\omega_1} \neq 0$ . Now, if  $(\mu_n)_{n \in \mathbb{N}}$  converges to  $\mu$ , let  $\nu_n = \mu_n - \sum_{\alpha \in S \setminus \{\omega_1\}} a_{\alpha} \delta_{\alpha}$  and notice that  $(\nu_n)_{n \in \mathbb{N}}$  is a bounded sequence in  $C(K)^*$  weak\* convergent to  $a_{\omega_1} \delta_{\omega_1}$ , contradicting Theorem 11. This concludes the proof that  $C$  is weak\* sequentially closed.

Finally, let us show that  $\delta_{\omega_1} \in \overline{C}^{w*}$ . Given  $\varepsilon > 0$ ,  $f_1, \dots, f_n \in C(K)$ , we have to

$$U = \bigcap_{i=1}^n f_i^{-1}[(f_i(\omega_1) - \varepsilon, f_i(\omega_1) + \varepsilon)]$$

is an open neighbourhood of  $\omega_1$ . Since  $\omega_1$  is an accumulation point of  $K$ , there is  $x \in K \setminus \{\omega_1\} \cap U$  and it follows that

$$\delta_x \in \{\mu \in C(K)^* : \forall 1 \leq i \leq n, |\mu(f_i) - \delta_{\omega_1}(f_i)| < \varepsilon\} \cap C.$$

Therefore  $\delta_{\omega_1} \in \overline{C}^{w*}$ .  $\square$

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